Take Home Assignment 1

1. A cylindrical container is to be manufactured with a volume of 200 cubic centimeters. The cylinder will be cut from sheets of stainless steel that cost $50.00/ m², and the caps will be cut from sheets of a different grade of stainless steel that cost $75.00/ m². Find the dimensions of the can that minimize the cost of the materials.

Find the rate of change \( dC/dV \) of the (minimal) materials-cost \( (C) \) of the container with respect to its volume \( (V) \).

If the height of the cylinder is \( h \) and the radius of the base is \( r \), both measured in cm, then the cost of materials is

\[
c = \frac{1}{100000} (100\pi hr + 150\pi r^2) = 0.01\pi hr + 0.015\pi r^2.
\]

The dimensions are constrained by the volume,

\[
\pi r^2 h = 200
\]

and the Lagrangian is therefore

\[
L(h, r, \lambda) = 0.01\pi hr + 0.015\pi r^2 - \lambda(\pi r^2 h - 200).
\]

The stationarity equations are

\[
\begin{align*}
L_h &= 0 \quad \Rightarrow \quad 0.01\pi r - \lambda \pi r^2 = 0 \\
L_r &= 0 \quad \Rightarrow \quad 0.01\pi h + 0.03\pi r - 2\lambda \pi rh = 0 \\
L_\lambda &= 0 \quad \Rightarrow \quad 200 - \pi r^2 h = 0
\end{align*}
\]

Solving the first two equations for \( \lambda \) gives

\[
\lambda = \frac{0.01}{r} = \frac{0.005}{r} + \frac{0.015}{h}.
\]

Clearing denominators and simplifying shows that

\[
h = 3r.
\]

Substituting for \( h \) in the constraint gives

\[
3\pi r^3 = 200 \quad \Rightarrow \quad r^* = \sqrt[3]{\frac{200}{3\pi}} \approx 2.77,
\]

which means that \( h^* = 3r^* = \sqrt[3]{\frac{1800}{\pi}} \approx 8.31 \) and the minimal materials cost is

\[
c^* = 0.01\pi h^* r^* + 0.015\pi (r^*)^2 \approx $2.53.
\]

The envelope theorem tells us that

\[
\frac{dc^*}{dV} = \lambda^* = \frac{0.01}{r^*} \approx 0.0036.
\]
2. Find the average distance to the origin of points in the ball

\[ x^2 + y^2 + z^2 \leq R^2. \]

The average distance to the origin of points in the ball of radius \( R \) centered at the origin, \( B_R \), is given by

\[
\frac{1}{\text{Vol}(B_R)} \iiint_{B_R} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz.
\]

This integral is easiest to compute in spherical coordinates,

\[
\iiint_{B_R} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \int_0^{2\pi} \int_0^\pi \int_0^R r \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi = \pi R^4
\]

so the average distance to the origin is

\[
\frac{\pi R^4}{\frac{4}{3} \pi R^3} = \frac{3}{4} R.
\]

3. Find the singular value decomposition of the matrix

\[ A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}. \]

First,

\[ AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \]

which has characteristic equation \( \lambda^2 - 34\lambda + 225 = 0 \) and eigenvalues \( \lambda_1 = 25 \) and \( \lambda_2 = 9 \), and corresponding orthonormal eigenvectors

\[ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \]

Next,

\[ A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}. \]

Recall that \( A^T A \) and \( AA^T \) have the same nonzero eigenvalues, so the eigenvalues of \( A^T A \) are \( \lambda_1 = 25 \), \( \lambda_2 = 9 \) and \( \lambda_3 = 0 \), with corresponding orthonormal eigenvectors

\[ v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \text{ } v_2 = \begin{bmatrix} \frac{1}{3 \sqrt{2}} \\ \frac{1}{3 \sqrt{2}} \\ \frac{2}{3 \sqrt{2}} \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}. \]

The singular values of \( A \) are \( \sigma_1 = \sqrt{25} = 5 \) and \( \sigma_2 = \sqrt{9} = 3 \) and singular value decomposition of \( A \) is

\[ A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3 \sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3 \sqrt{2}} & \frac{1}{3} \end{bmatrix}^T. \]

4. Find an orthogonal transformation of \( \mathbb{R}^3 \) that transforms the quadratic form

\[ Q(x, y, z) = x^2 + 2xy + 4xz + 2y^2 + 2yz + z^2 \]

to the diagonal form

\[ Q(u, v, w) = \alpha u^2 + \beta v^2 + \gamma w^2 \]

(and find the coefficients \( \alpha, \beta \) and \( \gamma \)).
The quadratic form $Q(x, y, z)$ may be written as

$$Q(x, y, z) = x^T A_Q x,$$

where

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad A_Q = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Since $A_Q$ is symmetric, we can find matrices $U$ and $D$ such that

$$A = U^T D U,$$

with

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad U = [u_1 \ u_2 \ u_3],$$

where $\lambda_1, \lambda_2$ and $\lambda_3$ are the eigenvalues of $A$ and $U$ is an orthogonal matrix with column $u_i$ being an eigenvector belonging to $\lambda_i$. Therefore, for $x \in \mathbb{R}^3$, we have

$$x^T A x = x^T (U^T D U) x = (U x)^T D (U x) = u^T D u = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2,$$

where

$$u = U x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

To find $D$ and $U$, we solve the characteristic equation of $A$:

$$\det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (1 - \lambda)^2 (4 - \lambda).$$

Thus the eigenvalues of $A$ are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.

To find the columns of $U$, we solve $(A - I)x = 0$ and $(A - 4I)x = 0$. First an orthogonal basis for the set of solutions of $(A - I)x = 0$ is given by

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

and an orthonormal basis is given by

$$u_1 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}.$$ 

The set of solutions to $(A - 4I)x = 0$ is spanned by the vector $x = [1 \ 1 \ 1]^T$, so

$$u_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$ 

Thus, the required orthogonal transformation of $\mathbb{R}^3$ is given by the matrix

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

and the diagonalized form is given by

$$Q(u, v, w) = u^2 + v^2 + 4w^2.$$
5. Find the unit tangent, normal and binormal, \( \hat{t}, \hat{n}, \hat{b} \), and the curvature \( \kappa \) as functions of \( t \) for the helix

\[ r(t) = a \cos(\omega t) \mathbf{i} + a \sin(\omega t) \mathbf{j} + bt \mathbf{k}. \]

First,

\[ \hat{t} = \frac{dr}{dt} = \frac{1}{\sqrt{a^2 \omega^2 + b^2}} (-a \omega \sin(\omega t) \mathbf{i} + a \omega \cos(\omega t) \mathbf{j} + b \mathbf{k}), \]

which also shows that

\[ \frac{ds}{dt} = \left| \frac{dr}{dt} \right| = \sqrt{a^2 \omega^2 + b^2}. \]

Next,

\[ \hat{n} = \frac{\hat{t}}{\left| \frac{d\hat{t}}{ds} \right|} = \frac{\frac{d\hat{t}}{dt}(dt/ds)}{\left| \frac{d\hat{t}}{dt} \right|} = \frac{\hat{t}}{\left| \frac{d\hat{t}}{dt} \right|} = -\cos(\omega t) \mathbf{i} - \sin(\omega t) \mathbf{j}, \]

where

\[ \left| \frac{d\hat{t}}{dt} \right| = \frac{a \omega^2}{\sqrt{a^2 \omega^2 + b^2}}. \]

For the binormal we have

\[ \hat{b} = \hat{t} \times \hat{n} = \frac{1}{\sqrt{a^2 \omega^2 + b^2}} (b \sin(\omega t) \mathbf{i} - b \cos(\omega t) \mathbf{j} + a \omega \mathbf{k}). \]

Finally, from

\[ \left| \frac{d\hat{t}}{dt} \right| = \frac{ds}{dt} \mid \frac{ds}{dt} \mid = \kappa \left| \frac{ds}{dt} \right| \]

we have

\[ \kappa = \left| \frac{d\hat{t}}{dt} \right| \left| \frac{ds}{dt} \right|^{-1} = \frac{a \omega^2}{a^2 \omega^2 + b^2}, \]

i.e., the helix has constant curvature for all \( t \), which is not surprising when you think about it.

6. A function \( \varphi(x, y, z) \) (a scalar field) is called radial if it is constant on spheres around the origin, i.e., \( \varphi(x, y, z) = \varphi(r) \), where \( r = \sqrt{x^2 + y^2 + z^2} \).

a. What is the Laplacian of a radial function? (Suggestion: use spherical coordinates).

In spherical coordinates the Laplacian is given by

\[ \nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}, \]

and so for a radial function (that does not depend on \( \theta \) and \( \phi \)), the Laplacian is simply

\[ \nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) = \frac{1}{r^2} \left( 2r \frac{\partial \varphi}{\partial r} + r^2 \frac{\partial^2 \varphi}{\partial r^2} \right) = \frac{2}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial r^2}. \]

b. A function \( u(x, y, z) \) is harmonic if \( \nabla^2 u = 0 \). Show that a radial harmonic function \( u(x, y, z) \) defined in all of \( \mathbb{R}^3 \) must be constant.

If \( u \) is radial and harmonic, then

\[ \frac{2}{r} u' + u'' = 0, \]

where \( u' \) and \( u'' \) are derivatives with respect to \( r \). It follows that either \( u' = 0 \) or

\[ \frac{u''}{u'} = -\frac{2}{r} \implies \ln |u'| = -2 \ln r + C \implies u' = \frac{A}{r^2}, \]
where $A = \pm e^C$. This would imply that $u = -\frac{A}{r} + k$, making $u$ undefined at the origin ($r = 0$), so it must be the case that $u' = 0$ and $u$ is constant.