Take Home Assignment 1

1. A cylindrical container is to be manufactured with a volume of 200 cubic centimeters. The cylinder will be cut from sheets of stainless steel that cost \$50.00/ m^2, and the caps will be cut from sheets of a different grade of stainless steel that cost \$75.00/ m^2. Find the dimensions of the can that minimize the cost of the materials.

Find the rate of change dC/dV of the (minimal) materials-cost (C) of the container with respect to its volume (V).

If the height of the cylinder is h and the radius of the base is r, both measured in cm, then the cost of materials is

$$c = \frac{1}{10000}\left(100\pi hr + 150\pi r^2\right) = 0.01\pi hr + 0.015\pi r^2.$$

The dimensions are constrained by the volume,

$$\pi r^2 h = 200$$

and the Lagrangian is therefore

$$L(h, r, \lambda) = 0.01\pi hr + 0.015\pi r^2 - \lambda(\pi r^2 h - 200).$$

The stationarity equations are

$$\begin{aligned} L_h = 0 \quad &\Rightarrow \quad 0.01\pi r - \lambda\pi r^2 = 0 \\ L_r = 0 \quad &\Rightarrow \quad 0.01\pi h + 0.03\pi r - 2\lambda\pi rh = 0 \\ L_\lambda = 0 \quad &\Rightarrow \quad 200 - \pi r^2 h = 0 \end{aligned}$$

Solving the first two equations for λ gives

$$\lambda = \frac{0.01}{r} = \frac{0.005}{r} + \frac{0.015}{h}.$$

Clearing denominators and simplifying shows that

$$h = 3r.$$

Substituting for h in the constraint gives

$$3\pi r^3 = 200 \implies r^* = \sqrt[3]{\frac{200}{3\pi}} \approx 2.77,$$

which means that $h^* = 3r^* = \sqrt[3]{\frac{1800}{\pi}} \approx 8.31$ and the minimal materials cost is

$$c^* = 0.01\pi h^* r^* + 0.015\pi (r^*)^2 \approx \$2.53.$$

The envelope theorem tells us that

$$\frac{dc^*}{dV} = \lambda^* = \frac{0.01}{r^*} \approx 0.0036.$$

2. Find the average distance to the origin of points in the ball

$$x^2 + y^2 + z^2 \le R^2.$$

The average distance to the origin of points in the ball of radius R centered at the origin, B_R, is given by

$$\frac{1}{\mathrm{Vol}(B_R)} \iiint_{B_R} \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz.$$

This integral is easiest to compute in spherical coordinates,

$$\iiint_{B_R} \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi = \pi R^4$$

so the average distance to the origin is

$$\frac{\pi R^4}{\frac{4}{3}\pi R^3} = \frac{3}{4} R.$$

3. Find the *singular value decomposition* of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

First,

$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

which has characteristic equation $\lambda^2 - 34\lambda + 225 = 0$ and eigenvalues $\lambda_1 = 25$ and $\lambda_2 = 9$, and corresponding orthonormal eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Next,

$$A^TA = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}.$$

Recall that A^TA and AA^T have the same nonzero eigenvalues, so the eigenvalues of A^TA are $\lambda_1 = 25$, $\lambda_2 = 9$ and $\lambda_3 = 0$, with corresponding orthonormal eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

The singular values of A are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{9} = 3$ and singular value decomposition of A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T$$

4. Find an orthogonal transformation of $\mathbb{R}^3$ that transforms the quadratic form

$$Q(x, y, z) = x^2 + 2xy + 4xz + 2y^2 + 2yz + z^2$$

to the diagonal form

$$\mathcal{Q}(u, v, w) = \alpha u^2 + \beta v^2 + \gamma w^2$$

(and find the coefficients α, β and γ).

The quadratic form $Q(x, y, z)$ may be written as

$$Q(x, y, z) = \mathbf{x}^T A_Q \mathbf{x},$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad A_Q = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Since A_Q is symmetric, we can find matrices U and D such that

$$A = U^T D U,$$

with

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix},$$

where λ_1, λ_2 and λ_3 are the eigenvalues of A and U is an orthogonal matrix with column $\mathbf{u}_i$ being an eigenvector belonging to λ_i. Therefore, for $\mathbf{x} \in \mathbb{R}^3$, we have

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (U^T D U)\mathbf{x} = (U\mathbf{x})^T D (U\mathbf{x}) = \mathbf{u}^T D \mathbf{u} = \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2,$$

where

$$\mathbf{u} = U\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

To find D and U, we solve the characteristic equation of A:

$$\det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (1 - \lambda)^2 (4 - \lambda).$$

Thus the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.

To find the columns of U, we solve $(A - I)\mathbf{x} = \mathbf{0}$ and $(A - 4I)\mathbf{x} = \mathbf{0}$. First an orthogonal basis for the set of solutions of $(A - I)\mathbf{x} = \mathbf{0}$ is given by

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

and an orthonormal basis is given by

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}.$$

The set of solutions to $(A - 4I)\mathbf{x} = \mathbf{0}$ is spanned by the vector $\mathbf{x} = [1\ 1\ 1]^T$, so

$$\mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Thus, the required orthogonal transformation of $\mathbb{R}^3$ is given by the matrix

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

and the diagonalized form is given by

$$\mathcal{Q}(u, v, w) = u^2 + v^2 + 4w^2.$$

5. Find the unit tangent, normal and binormal, $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$, and the curvature κ as functions of t for the helix

$$\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j} + bt\mathbf{k}.$$

First,

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \frac{1}{\sqrt{a^2\omega^2 + b^2}}\left(-a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j} + b\mathbf{k}\right),$$

which also shows that

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\omega^2 + b^2}.$$

Next,

$$\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}/ds}{|d\hat{\mathbf{t}}/ds|} = \frac{(d\hat{\mathbf{t}}/dt)(dt/ds)}{|(d\hat{\mathbf{t}}/dt)(dt/ds)|} = \frac{d\hat{\mathbf{t}}/dt}{|d\hat{\mathbf{t}}/dt|} = -\cos(\omega t)\mathbf{i} - \sin(\omega t)\mathbf{j},$$

where

$$\left|\frac{d\hat{\mathbf{t}}}{dt}\right| = \frac{a\omega^2}{\sqrt{a^2\omega^2 + b^2}}.$$

For the binormal we have

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \frac{1}{\sqrt{a^2\omega^2 + b^2}}\left(b\sin(\omega t)\mathbf{i} - b\cos(\omega t)\mathbf{j} + a\omega\mathbf{k}\right).$$

Finally, from

$$\left|\frac{d\hat{\mathbf{t}}}{dt}\right| = \left|\frac{d\hat{\mathbf{t}}}{ds}\right|\left|\frac{ds}{dt}\right| = \kappa\left|\frac{ds}{dt}\right|$$

we have

$$\kappa = \left|\frac{d\hat{\mathbf{t}}}{dt}\right|\left|\frac{ds}{dt}\right|^{-1} = \frac{a\omega^2}{a^2\omega^2 + b^2},$$

i.e., the helix has constant curvature for all t, which is not surprising when you think about it.

6. A function $\varphi(x, y, z)$ (a scalar field) is called *radial* if it is constant on spheres around the origin, i.e., $\varphi(x, y, z) = \varphi(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$.

a. What is the Laplacian of a radial function? (Suggestion: use spherical coordinates).

In Spherical coordinates the Laplacian is given by

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2},$$

and so for a radial function (that does *not* depend on θ and ϕ), the Laplacian is simply

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial\varphi}{\partial r} + r^2\frac{\partial^2\varphi}{\partial r^2}\right) = \frac{2}{r}\cdot\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2}.$$

b. A function $u(x, y, z)$ is *harmonic* if $\nabla^2 u = 0$. Show that a radial harmonic function $u(x, y, z)$ defined in all of $\mathbb{R}^3$ must be constant.

If u is radial and harmonic, then

$$\frac{2}{r}u' + u'' = 0,$$

where u' and u'' are derivatives with respect to r. It follows that either $u' = 0$ or

$$\frac{u''}{u'} = -\frac{2}{r} \implies \ln|u'| = -2\ln r + C \implies u' = \frac{A}{r^2},$$

where $A = \pm e^C$. This would imply that $u = -\frac{A}{r} + k$, making u undefined at the origin ($r = 0$), so it must be the case that $u' = 0$ and u is constant.