Take Home Assignment 2

1. Find the area of the region bounded by the curve $x^{2/3} + 9y^{2/3} = 4$.

Hint: Use a parametrization of form $x = \alpha \cos^n \vartheta$ and $y = \beta \sin^n \vartheta$ for the curve.

2. Consider the function $\varphi(x) = x^2 \cos(x/2)$ defined on the interval $[0, \pi]$. Write down the even and odd extensions of $\varphi(x)$ to the interval $[-\pi, \pi]$. Which of these two extensions will have the more quickly converging Fourier series (if either)? Why? Compute the Fourier coefficients for the extension whose series converges more rapidly.

3. Find the Fourier transform of the function $f(x) = e^{-a|x|}$, where $a > 0$. Use your answer (and the Fourier inversion formula) to compute

$$\int_0^\infty \frac{\cos(\omega x)}{\omega^2 + a^2}\, d\omega.$$

4. Use the residue theorem (and appropriately chosen contours in $\mathbb{C}$) to compute the integrals

$$I_1 = \int_0^\infty \frac{dx}{x^4 + 5x^2 + 4} \quad \text{and} \quad I_2 = \int_0^\infty \frac{\cos 3x\, dx}{x^4 + 5x^2 + 4}.$$

5. Use Rouché's theorem to show that the zeros of the polynomial $P(z) = 5z^5 + z^2 + z + 2$ all lie in the annulus

$$A = \left\{ z \in \mathbb{C} : \frac{13}{20} < |z| < 1 \right\}.$$

Brownie points for finding a narrower annulus (without using software to estimate the zeros!).