Take Home Assignment 2

1. Find the area of the region bounded by the curve \( x^{2/3} + 9y^{2/3} = 4 \).

   **Hint:** Use a parametrization of form \( x = \alpha \cos^n \vartheta \) and \( y = \beta \sin^n \vartheta \) for the curve.

   From Green’s theorem (see section 11.3) it follows that the area of the region bounded by a simple closed curve \( \gamma \) in the plane is given by the integral

   \[
   \frac{1}{2} \int_\gamma x \, dy - y \, dx.
   \]

   The given curve can be parametrized by \( x = 8 \cos^3 \vartheta \) and \( y = \frac{8}{27} \sin^3 \vartheta \) (with \( 0 \leq \vartheta \leq 2\pi \)), so \( dx = -24 \cos^2 \vartheta \sin \vartheta \, d\vartheta \) and \( dy = \frac{24}{27} \sin^2 \vartheta \cos \vartheta \, d\vartheta \). Hence the area of the specified region is given by

   \[
   A = \frac{1}{2} \int_\gamma x \, dy - y \, dx = \frac{96}{27} \int_0^{2\pi} \cos^2 \vartheta \sin^2 \vartheta \cos \vartheta \sin \vartheta \, d\vartheta
   = \frac{96}{27} \int_0^{2\pi} \cos^2 \vartheta \sin^2 \vartheta (\cos^2 \vartheta + \sin^2 \vartheta) \, d\vartheta
   = \frac{96}{27} \int_0^{2\pi} \cos^2 \vartheta \sin^2 \vartheta \, d\vartheta
   = \frac{96}{27} \int_0^{2\pi} \left( \frac{1}{2} \sin 2\vartheta \right)^2 \, d\vartheta
   = \frac{8}{9} \int_0^{2\pi} \sin^2 2\vartheta \, d\vartheta = \frac{8\pi}{9},
   \]

   using the identity \( \cos \vartheta \sin \vartheta = \frac{1}{2} \sin 2\vartheta \) and the fact that

   \[
   \int_0^{2\pi} \sin^2 2\vartheta \, d\vartheta = \int_0^{2\pi} \cos^2 2\vartheta \, d\vartheta,
   \]

   so \( \int_0^{2\pi} \sin^2 2\vartheta \, d\vartheta = \pi \).

2. Consider the function \( \varphi(x) = x^2 \cos(x/2) \) defined on the interval \([0, \pi]\). Write down the even and odd extensions of \( \varphi(x) \) to the interval \([-\pi, \pi]\). Which of these two extensions will have the more quickly converging Fourier series (if either)? Why? Compute the Fourier coefficients for the extension whose series converges more rapidly.

   The function \( \varphi(x) \) is an even function (on all of \( \mathbb{R} \)) so the even periodic extension of \( \varphi \) to \([-\pi, \pi]\) is given by \( \varphi \) itself:

   \[
   \varphi_E(x) = x^2 \cos(x/2) : -\pi \leq x \leq \pi,
   \]

   and the odd extension of \( \varphi \) is

   \[
   \varphi_O(x) = \begin{cases} 
   x^2 \cos(x/2) & : 0 \leq x \leq \pi \\
   -x^2 \cos(x/2) & : -\pi \leq x \leq 0 
   \end{cases}
   \]
Both the even and the odd periodic extensions are continuous on the whole line, since
\[ \varphi(\pi) = \varphi(-\pi) = 0. \]
Also, \( \varphi'(x) = 2x \cos(x/2) - \frac{1}{2}x^2 \sin(x/2) \), so both functions are differentiable at all \( x \neq n\pi \), and
\[ \varphi'_E(0) = 0 = \varphi'_O(0), \]
which means that both \( \varphi_E \) and \( \varphi_O \) are differentiable at all even multiples of \( \pi \). Finally at \( \pm \pi \), we have
\[ \varphi_E(\pi) = -\frac{\pi^2}{2} = \varphi_O(\pi), \]
but
\[ \varphi_O(-\pi) = -\frac{\pi^2}{2} \quad \text{while} \quad \varphi_E(-\pi) = \frac{\pi^2}{2}. \]
This means that \( \varphi_O \) is also differentiable at all odd multiples of \( \pi \), while \( \varphi_E \) is not.

Since \( \varphi_O \) is differentiable on all of \( \mathbb{R} \) and \( \varphi_E \) is not, the Fourier series for \( \varphi_O \) should converge more rapidly than that of \( \varphi_E \).

The Fourier series of the odd extension will include only \( \sin kx \) terms whose coefficients are given by
\[ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_O(x) \sin(kx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} \varphi_O(x) \sin(kx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(x/2) \sin(kx) \, dx, \]
since the integrand is even, being the product of two odd functions. To compute this integral, I’ll simplify the (trigonometric factor of the) integrand using the identity
\[ \sin(\alpha) \cos(\beta) = \frac{1}{2} \left( \sin(\alpha + \beta) + \sin(\alpha - \beta) \right). \]
Setting \( \alpha = kx \) and \( \beta = x/2 \), it follows that
\[
\begin{align*}
 b_k &= \frac{1}{\pi} \int_{0}^{\pi} x^2 \left( \sin \left( (k + \frac{1}{2}) x \right) + \sin \left( (k - \frac{1}{2}) x \right) \right) \, dx \\
 &= \frac{1}{\pi} \left( \int_{0}^{\pi} x^2 \sin \left( (k + \frac{1}{2}) x \right) \, dx + \int_{0}^{\pi} x^2 \sin \left( (k - \frac{1}{2}) x \right) \, dx \right) \\
 &= \frac{1}{\pi} \left( I_1(k) + I_2(k) \right)
\end{align*}
\]
To compute \( I_1(k) \) and \( I_2(k) \), I’ll use the formula

\[
\int x^2 \sin(\alpha x) \, dx = -\frac{1}{\alpha} x^2 \cos(\alpha x) + \frac{2}{\alpha} \int x \cos(\alpha x) \, dx \\
= -\frac{1}{\alpha} x^2 \cos(\alpha x) + \frac{2}{\alpha^2} x \sin(\alpha x) - \frac{2}{\alpha^2} \int \sin(\alpha x) \, dx \\
= -\frac{1}{\alpha} x^2 \cos(\alpha x) + \frac{2}{\alpha^2} x \sin(\alpha x) + \frac{2}{\alpha^3} \cos(\alpha x) + C
\]

Setting \( \alpha = \frac{2k + 1}{2} \), it follows that

\[
I_1(k) = -\frac{2}{2k + 1} x^2 \cos \left( \frac{2k+1}{2} x \right) + \frac{8}{(2k+1)^2} x \sin \left( \frac{2k+1}{2} x \right) + \frac{16}{(2k+1)^3} \cos \left( \frac{2k+1}{2} x \right) \bigg|_0^\pi \\
= \frac{(-1)^k 8\pi}{(2k+1)^2} - \frac{16}{(2k+1)^3}.
\]

Likewise, setting \( \alpha = \frac{2k - 1}{2} \), we have

\[
I_2(k) = -\frac{2}{2k - 1} x^2 \cos \left( \frac{2k-1}{2} x \right) + \frac{8}{(2k-1)^2} x \sin \left( \frac{2k-1}{2} x \right) + \frac{16}{(2k-1)^3} \cos \left( \frac{2k-1}{2} x \right) \bigg|_0^\pi \\
= \frac{(-1)^{k-1} 8\pi}{(2k-1)^2} - \frac{16}{(2k-1)^3}.
\]

It follows that

\[
b_k = \frac{1}{\pi} \left( \frac{(-1)^k 8\pi}{(2k+1)^2} - \frac{16}{(2k+1)^3} + \frac{(-1)^{k-1} 8\pi}{(2k-1)^2} - \frac{16}{(2k-1)^3} \right) \\
= \frac{(-1)^{k-1} 164k}{(2k-1)^2(2k+1)^2} - \frac{16}{\pi} \left( \frac{1}{(2k+1)^3} + \frac{1}{(2k-1)^3} \right) \\
= 64k \left( \frac{(-1)^{k-1}}{(2k-1)^2(2k+1)^2} - \frac{4k^2 + 3}{(2k+1)^3(2k-1)^3\pi} \right)
\]

The first four coefficients are

\[
b_1 \approx 1.8295, \quad b_2 \approx -0.7983 \quad b_3 \approx 0.1011 \quad \text{and} \quad b_4 \approx -0.0863,
\]

and the graphs of \( \varphi(x) \) (solid black line) and its Fourier series, truncated at four terms (dashed red line) are displayed in Figure 2 below.

**3.** Find the Fourier transform of the function \( f(x) = e^{-|x|} \), where \( a > 0 \). Use your answer (and the Fourier inversion formula) to compute

\[
\int_0^\infty \frac{\cos(\omega x)}{\omega^2 + a^2} \, d\omega.
\]
Figure 2: Graphs of $\varphi_O(x)$ and its Fourier series, truncated at four terms.

Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{(a-i\omega)x} \, dx + \int_{0}^{\infty} e^{-(a+i\omega)x} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{a - i\omega} - \frac{1}{-a - i\omega} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{\omega^2 + a^2}$$

From the Fourier inversion formula, we have

$$e^{-a|x|} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega^2 + a^2} \, d\omega = \frac{a}{\pi} \left( \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{\omega^2 + a^2} \, d\omega + i \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{\omega^2 + a^2} \, d\omega \right) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{\omega^2 + a^2} \, d\omega,$$

because the imaginary integral is 0 since the integrand there is odd. It follows that

$$\int_{0}^{\infty} \frac{\cos(\omega x)}{\omega^2 + a^2} \, d\omega = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{\omega^2 + a^2} \, d\omega = \pi \frac{\pi}{2a} e^{-a|x|}.$$ 

4. Use the residue theorem (and appropriately chosen contours in $\mathbb{C}$) to compute the integrals

$$I_1 = \int_{0}^{\infty} \frac{dx}{x^4 + 5x^2 + 4} \quad \text{and} \quad I_2 = \int_{0}^{\infty} \frac{\cos 3x \, dx}{x^4 + 5x^2 + 4}.$$ 

The integrands of both integrals are even functions, so

$$I_1 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 5x^2 + 4} \quad \text{and} \quad I_2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 3x \, dx}{x^4 + 5x^2 + 4}.$$ 

For both integrals, I’ll use contours $\gamma_R$, that go from $-R$ to $R$ on the real line, and then back again following the half-circle $\{Re^{i\vartheta} : 0 \leq \vartheta \leq \pi\}$, as depicted below.
First note that \( z^4 + 5z^2 + 4 = (z^2 + 4)(z^2 + 1) \), so the poles of both integrands are \( \pm 2i \) and \( \pm i \), and in the upper half plane, the only poles are at \( i \) and \( 2i \).

Next, if \( |z| = R > \sqrt{18} \), then

\[
|z^4 + 5z^2 + 4| \geq R^4 - 5R^2 - 4 = R^4 \left( 1 - \frac{5}{R^2} - \frac{4}{R^4} \right) > R^4 \left( 1 - \frac{9}{R^2} \right) > \frac{R^4}{2}.
\]

Hence, if \( R > \sqrt{18} \) and \( \gamma'_R \) is the semicircular portion of \( \gamma_R \), then

\[
\left| \int_{\gamma'_R} \frac{dz}{z^4 + 5z^2 + 4} \right| < \frac{2}{R^4} \cdot \pi R = \frac{2\pi}{R^3}
\]

and therefore

\[
\lim_{R \to \infty} \int_{\gamma'_R} \frac{dz}{z^4 + 5z^2 + 4} = 0.
\]

For \( I_2 \), we will also need the following observation: if \( z = x + iy \) and \( y > 0 \), then

\[
|e^{3iz}| = |e^{-3y}e^{3ix}| = e^{-3y} < 1,
\]

and therefore \( \lim_{R \to \infty} \int_{\gamma'_R} e^{3iz} \frac{dz}{z^4 + 5z^2 + 4} = 0 \) as well.

To compute \( I_1 \), we have

\[
\int_{-\infty}^{\infty} \frac{dx}{x^4 + 5x^2 + 4} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^4 + 5x^2 + 4} = \lim_{R \to \infty} \oint_{\gamma_R} \frac{dz}{(z - i)(z + i)(z - 2i)(z + 2i)} = 2\pi i(\text{Res}(i) + \text{Res}(2i)).
\]

Since both poles are simple in this case, we have

\[
\text{Res}(i) = \lim_{z \to i} \left( (z - i) \cdot \frac{1}{(z - i)(z + i)(z - 2i)(z + 2i)} \right) = \frac{1}{(2i)(-i)(3i)} = -\frac{i}{6}
\]

and

\[
\text{Res}(2i) = \lim_{z \to 2i} \left( (z - 2i) \cdot \frac{1}{(z - i)(z + i)(z - 2i)(z + 2i)} \right) = \frac{1}{(i)(3i)(4i)} = \frac{i}{12},
\]

so

\[
I_1 = \pi i \left( -\frac{i}{6} + \frac{i}{12} \right) = \frac{\pi}{12}.
\]
For $I_2$, note that
\[
\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^4 + 5x^2 + 4} \, dx = \int_{-\infty}^{\infty} \cos(3x) \, dx + i \int_{-\infty}^{\infty} \sin(3x) \, dx = \int_{-\infty}^{\infty} \cos(3x) \, dx,
\]
since the integrand of the imaginary integral is odd, so that integral is 0. Therefore
\[
\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^4 + 5x^2 + 4} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{3ix}}{x^4 + 5x^2 + 4} \, dx = \lim_{R \to \infty} \oint_{\gamma_R} \frac{e^{3iz}}{z-i}(z+i)(z-2i)(z+2i) = 2\pi i (\text{Res}(i) + \text{Res}(2i)).
\]
In this case, we have
\[
\text{Res}(i) = -\frac{i}{6} e^{-3} \quad \text{and} \quad \text{Res}(2i) = \frac{i}{12} e^{-6},
\]
so
\[
I_2 = \pi i \left( -\frac{i}{6} e^{-3} + \frac{i}{12} e^{-6} \right) = \frac{\pi}{12} \left( 2e^{-3} - e^{-6} \right).
\]

5. Use Rouche’s theorem to show that the zeros of the polynomial $P(z) = 5z^5 + z^2 + z + 2$ all lie in the annulus
\[
A = \left\{ z \in \mathbb{C} : \frac{13}{20} < |z| < 1 \right\}.
\]
Brownie points for finding a narrower annulus (without using software to estimate the zeros!).

If $|z| = 1$, then $|5z^5| = 5$ and
\[
|z^2 + z + 2| \leq |z|^2 + |z| + 2 = 4,
\]
so $|5z^5| > |z^2 + z + 2|$ on the unit circle $\{ z : |z| = 1 \}$. Hence, by Rouche’s theorem, $F(z) = z^5$ and $P(z) = F(z) + z^2 + z + 2$ have the same number of zeros in the unit disk, namely 5.

If $|z| = 13/20$, then
\[
|5z^5 + z^2 + z| \leq 5 \cdot \frac{13^5}{20^5} + \frac{13^2}{20^2} + \frac{13}{20} = \frac{5288465}{3200000} < 2,
\]
and once again, by Rouche’s theorem, $G(z) = 2$ and $P(z) = G(z) + 5z^5 + z^2 + z$ have the same number of zeros (namely none) inside the disk $\{ z : |z| \leq 13/20 \}$.

Conclusion: all the zeros of $P(z)$ are contained in $A$, as claimed.