Take Home Assignment 2

1. Find the area of the region bounded by the curve $x^{2/3} + 9y^{2/3} = 4$.

Hint: Use a parametrization of form $x = \alpha \cos^n \vartheta$ and $y = \beta \sin^n \vartheta$ for the curve.

From Green's theorem (see section 11.3) it follows that the area of the region bounded by a simple closed curve γ in the plane is given by the integral

$$\frac{1}{2}\oint_\gamma x\,dy - y\,dx.$$

The given curve can be parametrized by $x = 8\cos^3\vartheta$ and $y = \dfrac{8}{27}\sin^3\vartheta$ (with $0 \le \vartheta \le 2\pi$), so $dx = -24\cos^2\vartheta\sin\vartheta\,d\vartheta$ and $dy = \dfrac{24}{27}\sin^2\vartheta\cos\vartheta\,d\vartheta$. Hence the area of the specified region is given by

$$\begin{aligned}
A = \frac{1}{2}\oint_\gamma x\,dy - y\,dx &= \frac{96}{27}\int_0^{2\pi} \cos^4\vartheta\sin^2\vartheta + \cos^2\vartheta\sin^4\vartheta\,d\vartheta \\
&= \frac{96}{27}\int_0^{2\pi} \cos^2\vartheta\sin^2\vartheta(\cos^2\vartheta + \sin^2\vartheta)\,d\vartheta \\
&= \frac{96}{27}\int_0^{2\pi} \cos^2\vartheta\sin^2\vartheta\,d\vartheta \\
&= \frac{96}{27}\int_0^{2\pi} \left(\frac{1}{2}\sin 2\vartheta\right)^2 d\vartheta \\
&= \frac{8}{9}\int_0^{2\pi} \sin^2 2\vartheta\,d\vartheta = \frac{8\pi}{9},
\end{aligned}$$

using the identity $\cos\vartheta\sin\vartheta = \frac{1}{2}\sin 2\vartheta$ and the fact that

$$\int_0^{2\pi} \sin^2 2\vartheta\,d\vartheta = \int_0^{2\pi} \cos^2 2\vartheta\,d\vartheta,$$

so $\int_0^{2\pi} \sin^2 2\vartheta\,d\vartheta = \pi$.

2. Consider the function $\varphi(x) = x^2\cos(x/2)$ defined on the interval $[0, \pi]$. Write down the even and odd extensions of $\varphi(x)$ to the interval $[-\pi, \pi]$. Which of these two extensions will have the more quickly converging Fourier series (if either)? Why? Compute the Fourier coefficients for the extension whose series converges more rapidly.

The function $\varphi(x)$ is an even function (on all of $\mathbb{R}$) so the even periodic extension of φ to $[-\pi, \pi]$ is given by φ itself:

$$\varphi_E(x) = x^2\cos(x/2) \ : \ -\pi \le x \le \pi,$$

and the odd extension of φ is

$$\varphi_O(x) = \begin{cases} x^2\cos(x/2) & : \ 0 \le x \le \pi \\ -x^2\cos(x/2) & : \ -\pi \le x \le 0 \end{cases}$$

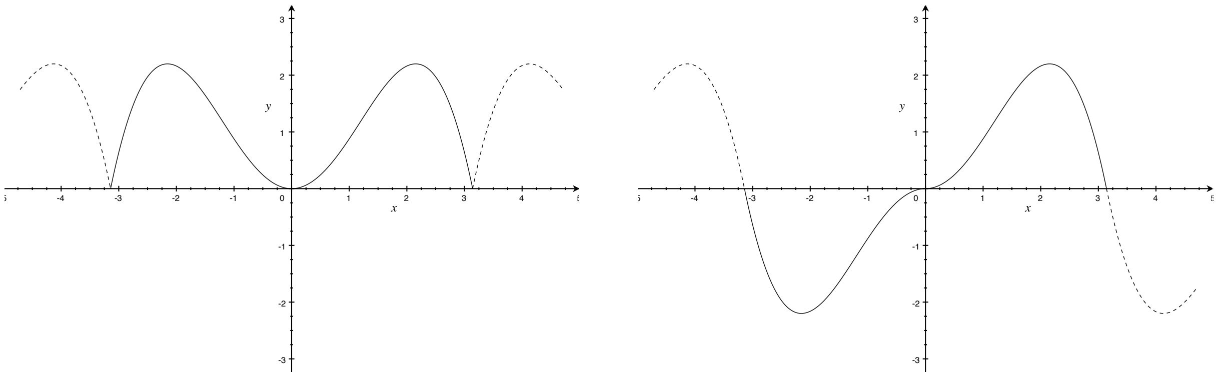


Figure 1: Graphs of $y = \varphi_E(x)$ and $y = \varphi_O(x)$

Both the even and the odd periodic extensions are continuous on the whole line, since

$$\varphi(\pi) = \varphi(-\pi) = 0.$$

Also, $\varphi'(x) = 2x\cos(x/2) - \frac{1}{2}x^2\sin(x/2)$, so both functions are differentiable at all $x \neq n\pi$, and

$$\varphi_E'(0) = 0 = \varphi_O'(0),$$

which means that both φ_E and φ_O are differentiable at all even multiples of π. Finally at $\pm\pi$, we have

$$\varphi_E(\pi) = -\frac{\pi^2}{2} = \varphi_O(\pi),$$

but

$$\varphi_O(-\pi) = -\frac{\pi^2}{2} \quad \text{while} \quad \varphi_E(-\pi) = \frac{\pi^2}{2}.$$

This means that φ_O is also differentiable at all odd multiples of π, while φ_E is *not*.

Since φ_O is differentiable on all of $\mathbb{R}$ and φ_E is not, the Fourier series for φ_O should converge more rapidly than that of φ_E.

The Fourier series of the odd extension will include only $\sin kx$ terms whose coefficients are given by

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi_O(x)\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} \varphi_O(x)\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(x/2)\sin(kx)\,dx,$$

since the integrand is even, being the product of two odd functions. To compute this integral, I'll simplify the (trigonometric factor of the) integrand using the identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right).$$

Setting $\alpha = kx$ and $\beta = x/2$, it follows that

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_0^{\pi} x^2\left(\sin\left(\left(k+\tfrac{1}{2}\right)x\right) + \sin\left(\left(k-\tfrac{1}{2}\right)x\right)\right)\,dx \\
&= \frac{1}{\pi}\left(\int_0^{\pi} x^2\sin\left(\left(k+\tfrac{1}{2}\right)x\right)\,dx + \int_0^{\pi} x^2\sin\left(\left(k-\tfrac{1}{2}\right)x\right)\,dx\right) \\
&= \frac{1}{\pi}\left(I_1(k) + I_2(k)\right)
\end{aligned}$$

To compute $I_1(k)$ and $I_2(k)$, I'll use the formula

$$\begin{aligned}\int x^2\sin(\alpha x)\,dx &= -\frac{1}{\alpha}x^2\cos(\alpha x)+\frac{2}{\alpha}\int x\cos(\alpha x)\,dx\\ &= -\frac{1}{\alpha}x^2\cos(\alpha x)+\frac{2}{\alpha^2}x\sin(\alpha x)-\frac{2}{\alpha^2}\int \sin(\alpha x)\,dx\\ &= -\frac{1}{\alpha}x^2\cos(\alpha x)+\frac{2}{\alpha^2}x\sin(\alpha x)+\frac{2}{\alpha^3}\cos(\alpha x)+C\end{aligned}$$

Setting $\alpha=\dfrac{2k+1}{2}$, it follows that

$$\begin{aligned}I_1(k) &= -\frac{2}{2k+1}x^2\cos\left(\tfrac{2k+1}{2}x\right)+\frac{8}{(2k+1)^2}x\sin\left(\tfrac{2k+1}{2}x\right)+\frac{16}{(2k+1)^3}\cos\left(\tfrac{2k+1}{2}x\right)\bigg|_0^{\pi}\\ &= \frac{(-1)^k 8\pi}{(2k+1)^2}-\frac{16}{(2k+1)^3}.\end{aligned}$$

Likewise, setting $\alpha=\dfrac{2k-1}{2}$, we have

$$\begin{aligned}I_2(k) &= -\frac{2}{2k-1}x^2\cos\left(\tfrac{2k-1}{2}x\right)+\frac{8}{(2k-1)^2}x\sin\left(\tfrac{2k-1}{2}x\right)+\frac{16}{(2k-1)^3}\cos\left(\tfrac{2k-1}{2}x\right)\bigg|_0^{\pi}\\ &= \frac{(-1)^{k-1} 8\pi}{(2k-1)^2}-\frac{16}{(2k-1)^3}.\end{aligned}$$

It follows that

$$\begin{aligned}b_k &= \frac{1}{\pi}\left(\frac{(-1)^k 8\pi}{(2k+1)^2}-\frac{16}{(2k+1)^3}+\frac{(-1)^{k-1}8\pi}{(2k-1)^2}-\frac{16}{(2k-1)^3}\right)\\ &= \frac{(-1)^{k-1}64k}{(2k-1)^2(2k+1)^2}-\frac{16}{\pi}\left(\frac{1}{(2k+1)^3}+\frac{1}{(2k-1)^3}\right)\\ &= 64k\left(\frac{(-1)^{k-1}}{(2k-1)^2(2k+1)^2}-\frac{4k^2+3}{(2k+1)^3(2k-1)^3\pi}\right)\end{aligned}$$

The first four coefficients are

$$b_1\approx 1.8295,\quad b_2\approx -0.7983\quad b_3\approx 0.1011\ \text{ and }\ b_4\approx -0.0863,$$

and the graphs of $\varphi_O(x)$ (solid black line) and its Fourier series, truncated at four terms (dashed red line) are displayed in Figure 2 below.

3. Find the Fourier transform of the function $f(x)=e^{-a|x|}$, where $a>0$. Use your answer (and the Fourier inversion formula) to compute

$$\int_0^{\infty}\frac{\cos(\omega x)}{\omega^2+a^2}\,d\omega.$$

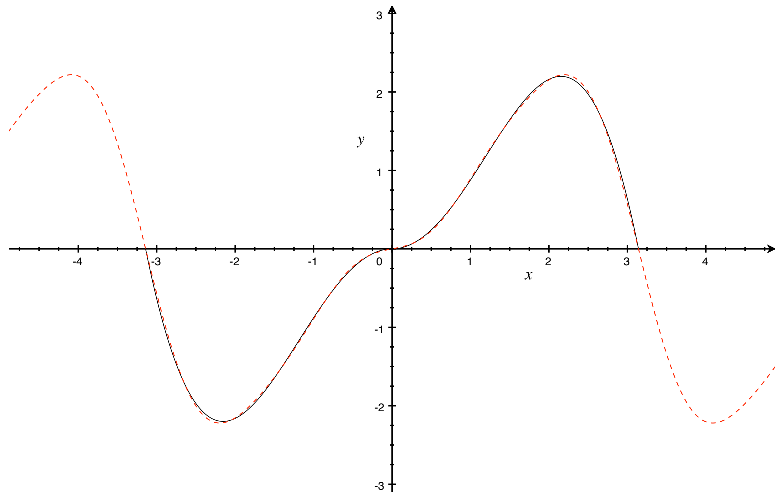


Figure 2: Graphs of $\varphi_O(x)$ and its Fourier series, truncated at four terms.

Fourier transform:

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{-i\omega x}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(a-i\omega)x}\,dx + \int_{0}^{\infty} e^{-(a+i\omega)x}\,dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a-i\omega} - \frac{1}{-a-i\omega}\right] \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{2a}{\omega^2+a^2}\end{aligned}$$

From the Fourier inversion formula, we have

$$e^{-a|x|} = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{\omega^2+a^2}\,d\omega = \frac{a}{\pi}\left(\int_{-\infty}^{\infty}\frac{\cos(\omega x)}{\omega^2+a^2}\,d\omega + i\int_{-\infty}^{\infty}\frac{\sin(\omega x)}{\omega^2+a^2}\,d\omega\right) = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\omega x)}{\omega^2+a^2}\,d\omega,$$

because the imaginary integral is 0 since the integrand there is odd. It follows that

$$\int_0^{\infty}\frac{\cos(\omega x)}{\omega^2+a^2}\,d\omega = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(\omega x)}{\omega^2+a^2}\,d\omega = \frac{\pi}{2a}e^{-a|x|}.$$

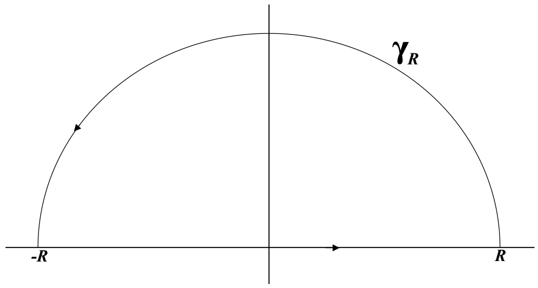
4. Use the residue theorem (and appropriately chosen contours in $\mathbb{C}$) to compute the integrals

$$I_1 = \int_0^{\infty}\frac{dx}{x^4+5x^2+4} \qquad \text{and} \qquad I_2 = \int_0^{\infty}\frac{\cos 3x\,dx}{x^4+5x^2+4}.$$

The integrands of both integrals are even functions, so

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+5x^2+4} \qquad \text{and} \qquad I_2 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos 3x\,dx}{x^4+5x^2+4}.$$

For both integrals, I'll use contours γ_R, that go from $-R$ to R on the real line, and then back again following the half-circle $\{Re^{i\vartheta} : 0 \le \vartheta \le \pi\}$, as depicted below.



First note that $z^4+5z^2+4=(z^2+4)(z^2+1)$, so the poles of both integrands are $\pm 2i$ and $\pm i$, and in the upper half plane, the only poles are at i and $2i$.

Next, If $|z|=R>\sqrt{18}$, then

$$|z^4+5z^2+4| \geq R^4-5R^2-4 = R^4\left(1-\frac{5}{R^2}-\frac{4}{R^4}\right) > R^4\left(1-\frac{9}{R^2}\right) > \frac{R^4}{2}.$$

Hence, if $R>\sqrt{18}R$ and γ_R' is the semicircular portion of γ_R, then

$$\left|\int_{\gamma_R'} \frac{dz}{z^4+5z^2+4}\right| < \frac{2}{R^4}\cdot \pi R = \frac{2\pi}{R^3}$$

and therefore

$$\lim_{R\to\infty}\int_{\gamma_R'} \frac{dz}{z^4+5z^2+4} = 0.$$

For I_2, we will also need the following observation: if $z=x+iy$ and $y>0$, then

$$|e^{3iz}| = |e^{-3y}e^{3ix}| = e^{-3y} < 1,$$

and therefore $\displaystyle\lim_{R\to\infty}\int_{\gamma_R'} \frac{e^{3iz}\,dz}{z^4+5z^2+4} = 0$ as well.

To compute I_1, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^4+5x^2+4} &= \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^4+5x^2+4} \\ &= \lim_{R\to\infty}\oint_{\gamma_R} \frac{dz}{(z-i)(z+i)(z-2i)(z+2i)} = 2\pi i(\mathrm{Res}(i)+\mathrm{Res}(2i)).\end{aligned}$$

Since both poles are simple in this case, we have

$$\mathrm{Res}(i) = \lim_{z\to i}\left((z-i)\cdot\frac{1}{(z-i)(z+i)(z-2i)(z+2i)}\right) = \frac{1}{(2i)(-i)(3i)} = -\frac{i}{6}$$

and

$$\mathrm{Res}(2i) = \lim_{z\to 2i}\left((z-2i)\cdot\frac{1}{(z-i)(z+i)(z-2i)(z+2i)}\right) = \frac{1}{(i)(3i)(4i)} = \frac{i}{12},$$

so

$$I_1 = \pi i\left(-\frac{i}{6}+\frac{i}{12}\right) = \frac{\pi}{12}.$$

For I_2, note that

$$\int_{-\infty}^{\infty} \frac{e^{3ix}\,dx}{x^4+5x^2+4} = \int_{-\infty}^{\infty} \frac{\cos(3x)\,dx}{x^4+5x^2+4} + i\int_{-\infty}^{\infty} \frac{\sin(3x)\,dx}{x^4+5x^2+4} = \int_{-\infty}^{\infty} \frac{\cos(3x)\,dx}{x^4+5x^2+4},$$

since the integrand of the imaginary integral is odd, so that integral is 0. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(3x)\,dx}{x^4+5x^2+4} &= \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{3ix}\,dx}{x^4+5x^2+4} \\ &= \lim_{R\to\infty}\oint_{\gamma_R} \frac{e^{3iz}\,dz}{(z-i)(z+i)(z-2i)(z+2i)} = 2\pi i(\mathrm{Res}(i)+\mathrm{Res}(2i)).\end{aligned}$$

In this case, we have

$$\mathrm{Res}(i) = -\frac{i}{6}e^{-3} \quad \text{and} \quad \mathrm{Res}(2i) = \frac{i}{12}e^{-6},$$

so

$$I_2 = \pi i\left(-\frac{i}{6}e^{-3} + \frac{i}{12}e^{-6}\right) = \frac{\pi}{12}\left(2e^{-3} - e^{-6}\right).$$

5. Use Rouché's theorem to show that the zeros of the polynomial $P(z) = 5z^5 + z^2 + z + 2$ all lie in the annulus

$$A = \left\{z \in \mathbb{C} : \frac{13}{20} < |z| < 1\right\}.$$

Brownie points for finding a narrower annulus (without using software to estimate the zeros!).

If $|z| = 1$, then $|5z^5| = 5$ and

$$\left|z^2 + z + 2\right| \le \left|z^2\right| + |z| + 2 = 4,$$

so $|5z^5| > |z^2 + z + 2|$ on the unit circle $\{z : |z| = 1\}$. Hence, by Rouché's theorem, $F(z) = z^5$ and $P(z) = F(z) + z^2 + z + 2$ have the same number of zeros in the unit disk, namely 5.

If $|z| = 13/20$, then

$$\left|5z^5 + z^2 + z\right| \le 5\cdot\frac{13^5}{20^5} + \frac{13^2}{20^2} + \frac{13}{20} = \frac{5288465}{3200000} < 2,$$

and once again, by Rouché's theorem, $G(z) = 2$ and $P(z) = G(z) + 5z^5 + z^2 + z$ have the same number of zeros (namely none) inside the disk $\{z : |z| \le 13/20\}$.

Conclusion: all the zeros of $P(z)$ are contained in A, as claimed.