Review Questions for Final Exam

1. (a) From first principles, find an *integrating factor* $\mu(x)$ for the general first order linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x).$$

(b) Solve the initial value problems

i. $\sin x \dfrac{dy}{dx} - 2\cos x\, y = \sin^3 x; \quad y(\pi/4) = 0.$

ii. $\dfrac{dy}{dx} + 2xy = 3xy^3; \quad y(0) = 1.$ (This one needs a substitution to make it linear.)

2. (a) Use the definition to find the Laplace transforms of $h(x) = H(x-2) - H(x-4)$, where $H(x)$ is the *Heaviside* function

$$H(x) = \begin{cases} 1 & : \quad x \geq 0 \\ 0 & : \quad x < 0 \end{cases}$$

(b) Use the Laplace transform method to solve the initial value problem

$$y'' + 2y' - 3y = H(x) - H(x-1); \quad y(0) = 1,\ y'(0) = 0.$$

3. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\cos(2t); \quad y(0) = y'(0) = 0.$$

4. Use *Green's* theorem to evaluate the integral

$$\oint_C x^2 y\, dx + 2xy^2\, dy,$$

where C is the triangle in $\mathbb{R}^2$ with corners $(0,0), (0,2)$ and $(1,4)$.

5. Compute the surface integral $\displaystyle\int_S x^2 + 2y^2\, dS$ over the surface

$$S = \{(x,y,z) : x^2 + y^2 = z^2 \ \text{ and } \ 0 \leq z \leq 1\}.$$

Suggestion: Use the parametrization $\mathbf{r} = \rho\cos\theta\mathbf{i} + \rho\sin\theta\mathbf{j} + \rho\mathbf{k}$ for the cone, with $0 \leq \rho \leq 1$ and $0 \leq \theta < 2\pi$.

6. If $\varphi(x,y,z)$ is a scalar field and $\mathbf{v}(x,y,z)$ is a vector field, show that

$$\nabla \times (\nabla\varphi) = 0 \ \text{ and } \ \nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

7. Suppose that $\mathbf{u} = (\phi(x,y), \psi(x,y), 0)$ is a continuous vector field confined to $\mathbb{R}^2$ that is both *solenoidal* and *irrotational*. Show that the function $f(x+iy) = \psi(x,y) + i\phi(x,y)$ is analytic in $\mathbb{C}$.

8. A (left) stochastic matrix $\mathbf{A}$ is an $n \times n$ matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

with nonnegative coefficients whose column-sums are all 1, i.e., for which $\displaystyle\sum_{i=1}^{n} a_{ij} = 1$ for each j.

(a) Show that a stochastic matrix always has an eigenvalue equal to 1.

(b) Find the eigenvalues and corresponding eigenvectors of the *stochastic* matrix $\mathbf{A} = \begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix}$.

(c) Show that if $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, then there is a vector $\mathbf{w}$ *that depends only on* $a + b$ such that $\mathbf{A}^n\mathbf{u} \to \mathbf{w}$.

9. In a large forest, foxes prey on rabbits while the rabbits feed on the (unlimited) vegetation. The change over time of the fox and rabbit populations in the forest is modeled by the following linear system:

$$\begin{bmatrix} F_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -p & 1.2 \end{bmatrix} \cdot \begin{bmatrix} F_k \\ R_k \end{bmatrix},$$

where F_k is the size of the fox population in year k, R_k is the size of the rabbit population in year k and p is a positive number called the *predation parameter*, that accounts for deaths in the rabbit population due to predation by foxes. The matrix, $\mathbf{T}_p$, on the right hand side of the equation is called the *transition matrix* of the model.

(i) Find the eigenvalues and corresponding eigenvectors for the transition matrix when the predation parameter is $p = 0.275$.

(ii) If $F_0 = 4$, $R_0 = 20$ and $p = 0.275$, what can you say about the limit $\lim_{k\to\infty} \frac{R_k}{F_k}$?

(iii) With $p = 0.275$, find the *critical ratio* ρ^* such that if $R_0/F_0 > \rho^*$, then both populations survive, and if $R_0/F_0 \leq \rho^*$, then both populations die off. Explain your work.

Hint for (iv): *Cramer's rule* is useful here.

(iv) Show that if (the predation parameter) $49/120 > p > 1/3$, then both populations die off (rapidly) if $F_0 > 0$, regardless of R_0. What happens when $F_0 = 0$? What happens when $p \geq 49/120$?

10. Consider the function $f(x) = e^x$ defined on the interval $[0, 1]$.

(a) Sketch the graph of its periodic extension to $\mathbb{R}$ with period 1, as well as its even and odd periodic extensions to $\mathbb{R}$ with period 2.

(b) Which of these periodic extensions will yield the *best* Fourier series expansion? Why?

(c) Compute the Fourier coefficients for all three periodic extensions. Were you right?

11. Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$ be a non constant polynomial with complex coefficients (i.e., $n > 0$ and $a_n \neq 0$). Use Rouché's Theorem to show that $p(z)$ has exactly n roots in $\mathbb{C}$ (counting multiplicity).

12. Use Cauchy's theorem and the countours γ_r (illustrated below) in $\mathbb{C}$ to show that

$$\int_0^\infty \cos(\alpha x^2)\, dx = \sqrt{\frac{\pi}{8\alpha}}.$$