Review Questions for Final Exam – Solutions

1. (a) From first principles, find an *integrating factor* $\mu(x)$ for the general first order linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x).$$

Solution.
We want to find $\mu(x)$ *such that*

$$\mu(x)(y' + p(x)y) = \frac{d}{dx}(\mu(x)y).$$

This holds if and only if $\dfrac{d\mu}{dx} = p(x)\mu(x)$, *which in turn implies that*

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = p(x)$$

and integrating both sides leads to

$$\ln(\mu(x)) = \int p(x)\,dx,$$

so $\mu(x) = \exp(\int p(x)\,dx)$.

(b) Solve the initial value problems

i. $\sin x \dfrac{dy}{dx} - 2\cos x\, y = \sin^3 x; \quad y(\pi/4) = 0.$

Solution. *First divide through by* $\sin x$*:*

$$\frac{dy}{dx} - 2\frac{\cos x}{\sin x}\,y = \sin^2 x. \tag{1}$$

Now, find the integrating factor:

$$\mu(x) = \exp\left(-2\int \frac{\cos x}{\sin x}\,dx\right) = \exp(-2\ln(\sin x)) = \sin^{-2} x.$$

Next, multiply equation (1) *through by* $\sin^{-2} x$*:*

$$\sin^{-2} x\left(\frac{dy}{dx} - 2\frac{\cos x}{\sin x}\,y\right) = \frac{d}{dx}\left(y\sin^{-2} x\right) = 1.$$

Integrate and solve for y*:*

$$y\sin^{-2} x = x + C \implies y = (x + C)\sin^2 x.$$

Finally, use the boundary condition to solve for C*:*

$$0 = y(\pi/4) = (\pi/4 + C)\sin^2(\pi/4) = \frac{\pi}{4} + C \implies C = -\frac{\pi}{4},$$

so

$$y = (x - \pi/4)\sin^2 x.$$

ii. $\dfrac{dy}{dx} + 2xy = 3xy^3; \quad y(0) = 1$. (This one needs a substitution to make it linear.)

Solution. *As alluded to, this is a Bernoulli equation, and the substitution* $u = y^{-2}$*, which implies that* $\dfrac{dy}{dx} = -\frac{1}{2}y^3\dfrac{du}{dx}$*, transforms the original equation to*

$$\frac{du}{dx} - 4xu = -6x, \tag{2}$$

after multiplication by $-2y^{-3}$*. The integrating factor for* (2) *is*

$$\mu(x) = \exp\left(-4\int x\,dx\right) = e^{-2x^2},$$

and multiplying (2) *through by this factor gives*

$$e^{-2x^2}\left(\frac{du}{dx} - 4xu\right) = \frac{d}{dx}\left(ue^{-2x^2}\right) = -6xe^{-2x^2}.$$

Integrating both sides and solving for u*, we have*

$$ue^{-2x^2} = \frac{6}{4}\int -4xe^{-2x^2}\,dx = \frac{3}{2}e^{-2x^2} + C \implies u = \frac{3}{2} + Ce^{2x^2}.$$

The boundary condition $y(0) = 1$ *implies that* $u(0) = 1^{-2} = 1$*, and using this to solve for* C*, we have*

$$1 = \frac{3}{2} + C \implies C = -\frac{1}{2},$$

so $u = \frac{1}{2}(3 - e^{2x^2})$*, and*

$$y = u^{-1/2} = \sqrt{\frac{2}{3 - e^{2x^2}}}.$$

Comment: *The last equation can be rewritten as* $y' = x(2 - 3y^3)$ *and the function on the right is continuously differentiable with respect to both variables in the entire* xy*-plane (which is an infinite rectangle centered at* $(0, 1)$*). Nonetheless, the solution we found is only defined in the interval* $\left(-\sqrt{\frac{\ln 3}{2}}, \sqrt{\frac{\ln 3}{2}}\right) \approx (-0.741, 0.741)$ *around* 0.

2. (a) Use the definition to find the Laplace transforms of $h(x) = H(x-2) - H(x-4)$, where $H(x)$ is the *Heaviside* function

$$H(x) = \begin{cases} 1 & : \ x \geq 0 \\ 0 & : \ x < 0 \end{cases}$$

Solution.

$$\mathcal{L}(h(x)) = \int_0^\infty (H(x-2) - H(x-4))e^{-xs}\,dx = \int_2^4 e^{-xs}\,dx = -\frac{1}{s}e^{-xs}\bigg|_2^4 = \frac{e^{-2s} - e^{-4s}}{s}.$$

(b) Use the Laplace transform method to solve the initial value problem

$$y'' + 2y' - 3y = H(x) - H(x-1); \quad y(0) = 1,\ y'(0) = 0.$$

Solution. *The idea is to apply the Laplace transform to both sides of the differential equation. This incorporates the boundary conditions because:*

$$\mathcal{L}(y') = sY(s) - y(0) \quad \textit{and} \quad \mathcal{L}(y'') = s^2Y(s) - y(0)s - y'(0),$$

where $Y(s) = \mathcal{L}(y)$. *We then solve the resulting* ***algebraic*** *equation for* $Y(s)$ *and use the inverse Laplace transform to find* y.[†]

In this case, we have

$$\begin{aligned}\mathcal{L}(y'' + 2y' - 3y) = \mathcal{L}(H(x) - H(x-1)) &\implies s^2Y(s) - s + 2sY(s) - 2 - 3Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} \\ &\implies Y(s)(s^2 + 2s - 3) = \frac{1 - e^{-s}}{s} + s + 2\end{aligned}$$

Therefore,

$$\begin{aligned}Y(s) &= \frac{s^2 + 2s + 1}{s(s-1)(s+3)} - e^{-s}\frac{1}{s(s-1)(s+3)} \\ &= -\frac{1/3}{s} + \frac{1}{s-1} + \frac{1/3}{s+3} - \frac{1/3e^{-s}}{s} + \frac{1/4e^{-s}}{s-1} + \frac{1/12e^{-s}}{s+3}\end{aligned}$$

From section 13.2.2 in the book, we learn that

$$\mathcal{L}^{-1}\left(e^{-bs}F(s)\right) = \begin{cases} 0 & : \; 0 < x \leq b \\ f(x-b) & : \; x > b \end{cases} = f(x-b)H(x-b)$$

where $f(x) = \mathcal{L}^{-1}(F(s))$. *This means that*

$$\begin{aligned}y &= \mathcal{L}^{-1}\left(-\frac{1/3}{s} + \frac{1}{s-1} + \frac{1/3}{s+3} - \frac{1/3e^{-s}}{s} + \frac{1/4e^{-s}}{s-1} + \frac{1/12e^{-s}}{s+3}\right) \\ &= -\frac{1}{3} + e^x + \frac{1}{3}e^{-3x} - \frac{1}{3}H(x-1) + \frac{1}{4}e^{x-1}H(x-1) + \frac{1}{12}e^{-3(x-1)}H(x-1)\end{aligned}$$

3. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\cos(2t); \quad y(0) = y'(0) = 0. \tag{3}$$

Solution. *Two methods...*

(i) Undetermined coefficients: *First find the general solution of the complementary (homogeneous) equation*

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \tag{4}$$

Characteristic equation — $r^2 + 2r + 5 = 0 \implies r = \dfrac{-2 \pm \sqrt{4 - 20}}{2} \implies r = -1 \pm 2i$, *so a basis for the space of solutions of equation* (4) *is given by*

$$u_1(t) = e^{(-1+2i)t} \quad \textit{and} \quad u_2(t) = e^{(-1-2i)t},$$

[†] In the most common cases, we can 'read' the inverse transform from a table.

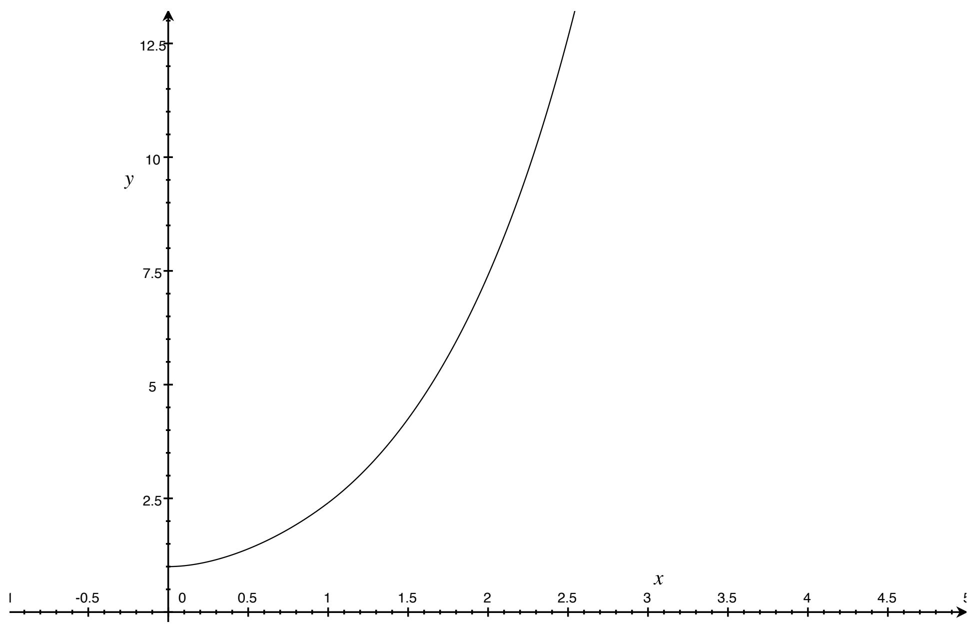


Figure 1: Graph of the solution of $y'' + 2y' - 3y = H(x) - H(x-1);\ y(0) = 1,\ y'(0) = 0.$

and a basis of ***real-valued*** *solutions is given by*

$$y_1 = \frac{1}{2}(u_1 + u_2) = e^{-t}\cos 2t \quad \textit{and} \quad y_2 = \frac{1}{2i}(u_1 - u_2) = e^{-t}\sin 2t.$$

Thus, the general solution of equation (4) *is*

$$y_h = c_1 y_1 + c_2 y_2 = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Next, find a particular solution of the differential equation (3) *using the method of* ***undetermined coefficients****, i.e., look for a solution of the form* $y_p = A\cos 2t + B\sin 2t$*. We have* $y_p' = -2A\sin 2t + 2B\cos 2t$ *and* $y_p'' = -4A\cos 2t - 4B\sin 2t$*, and substituting these into the original equation gives*

$$y_p'' + 2y_p' + 5y_p = \cos 2t(-4A + 4B + 5A) + \sin 2t(-4B - 4A + 5B) = 3\cos 2t,$$

and leads to the pair of linear equations

$$\begin{aligned} A + 4B &= 3 \\ -4A + B &= 0 \end{aligned}$$

with solution

$$A = \frac{\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix}} = \frac{3}{17} \qquad \textit{and} \qquad B = \frac{\begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix}} = \frac{12}{17}$$

It follows that the general solution of (3) *is*

$$y = y_p + y_h = \frac{3}{17}\cos 2t + \frac{12}{17}\sin 2t + e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Finally, we use the initial conditions, $y(0) = y'(0) = 0$ to determine c_1 and c_2. First, we have

$$0 = y_p(0) = \frac{3}{17} + c_1 \implies c_1 = -\frac{3}{17}.$$

Second, $y_p' = -\frac{6}{17}\sin 2t + \frac{24}{17}\cos 2t + e^{-t}\left((2c_2 - c_1)\cos 2t - (2c_1 + c_2)\sin 2t\right)$, *so*

$$0 = y_p'(0) = \frac{24}{17} + 2c_2 - c_1 = 2c_2 + \frac{27}{17} \implies c_2 = -\frac{27}{34},$$

and the solution to the initial value problem is

$$y = \frac{3}{17}\cos 2t + \frac{12}{17}\sin 2t - \frac{1}{34}e^{-t}(6\cos 2t + 27\sin 2t).$$

(ii) Laplace transform: *First take Laplace transforms of both sides of equation (3), using the given boundary conditions*

$$\mathcal{L}\left(\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right) = \mathcal{L}(3\cos(2t)) \implies s^2Y(s) + 2sY(s) + 5Y(s) = \frac{3s}{s^2+4},$$

where $Y(s)$ is the Laplace transform of the (as-of-yet unknown) solution y. Next, solve the equation above for $Y(s)$:

$$Y(s)(s^2 + 2s + 5) = \frac{3s}{s^2+4} \implies Y(s) = \frac{3s}{(s^2+4)(s^2+2s+5)}.$$

Now use a partial fraction decomposition of the rational function on the right in order to more easily identify the inverse Laplace transform. Note that both quadratic factors in the denominator cannot be factored over the real numbers, and this leads to a partial fraction decomposition of the form

$$\begin{aligned}
\frac{3s}{(s^2+4)(s^2+2s+5)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+2s+5} \\
&= \frac{(As+B)(s^2+2s+5) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+2s+5)} \\
&= \frac{(A+C)s^3 + (2A+B+D)s^2 + (5A+2B+4C)s + (5B+4D)}{(s^2+4)(s^2+2s+5)}.
\end{aligned}$$

Comparing the coefficients of s^3, s^2, s and the constant coefficient on both sides of the equation above, yields a system of four linear equations in the unknown coefficients A, B, C and D:

$$\begin{aligned}
A + C &= 0 \\
2A + B + D &= 0 \\
5A + 2B + 4C &= 3 \\
5B + 4D &= 0
\end{aligned}$$

The first and fourth equation show that $C = -A$ and $D = -\frac{5}{4}B$, and substituting these expressions in the second and third equation gives the pair of equations

$$\begin{aligned}
2A - \frac{1}{4}B &= 0 \\
A + 2B &= 3
\end{aligned}$$

with solution

$$A = \frac{\begin{vmatrix} 0 & -1/4 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1/4 \\ 1 & 2 \end{vmatrix}} = \frac{3/4}{17/4} = \frac{3}{17} \qquad \textit{and} \qquad B = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1/4 \\ 1 & 2 \end{vmatrix}} = \frac{6}{17/4} = \frac{24}{17}.$$

It follows that $C = -\frac{3}{17}$ *and* $D = -\frac{30}{17}$*, and therefore*

$$\begin{aligned} Y(s) &= \frac{3s}{(s^2+4)(s^2+2s+5)} \\ &= \frac{\frac{3}{17}s + \frac{24}{17}}{s^2+4} - \frac{\frac{3}{17}s + \frac{30}{17}}{s^2+2s+5} \\ &= \frac{3}{17} \cdot \frac{s}{s^2+4} + \frac{12}{17} \cdot \frac{2}{s^2+4} - \frac{3}{17} \cdot \frac{s+1}{(s+1)^2+4} - \frac{27}{34} \cdot \frac{2}{(s+1)^2+4}. \end{aligned}$$

Note that the coefficients of the rational functions in the second row have been distributed so that the rational functions in the third row all have the form

$$\frac{s+a}{(s+a)^2+b^2} \qquad \textit{or} \qquad \frac{b}{(s+a)^2+b^2},$$

whose inverse Laplace transforms are

$$e^{-at}\cos(bt) \qquad \textit{and} \qquad e^{-at}\sin(bt)$$

respectively. It follows that the solution is

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{3}{17} \cdot \frac{s}{s^2+4} + \frac{12}{17} \cdot \frac{2}{s^2+4} - \frac{3}{17} \cdot \frac{s+1}{(s+1)^2+4} - \frac{27}{34} \cdot \frac{2}{(s+1)^2+4}\right)$$

which is (not surprisingly) the same solution we found the first time around.

Conclusion: *The algebra is unavoidable, so pick your poison.*

4. Use *Green's* theorem to evaluate the integral

$$\oint_C x^2 y \, dx + 2xy^2 \, dy,$$

where C is the triangle in $\mathbb{R}^2$ with corners $(0,0), (0,2)$ and $(1,4)$.

Solution. *Setting* $P = x^2 y$ *and* $Q = 2xy^2$*, we have* $P_y = x^2$ *and* $Q_x = 2y^2$*. Invoking Green's theorem, we have*

$$\oint_C x^2 y \, dx + 2xy^2 \, dy = \int_T 2y^2 - x^2 \, dx \, dy,$$

where T *is the triangular region bounded by* C*:*

$$T = \{(x,y) : 0 \le x \le 1 \textit{ and } 4x \le y \le 2x+2\}.$$

It follows that

$$
\begin{aligned}
\oint_C x^2y\,dx + 2xy^2\,dy &= \int_0^1 \int_{4x}^{2x+2} 2y^2 - x^2\,dy\,dx \\
&= \int_0^1 \frac{2}{3}\left((2x+2)^3 - (4x)^3\right) - x^2(2x+2-4x)\,dx \\
&= \frac{2}{3}\int_0^1 -53x^3 + 21x^2 + 24x + 8\,dx \\
&= \frac{2}{3}\left(-\frac{53}{4} + 7 + 12 + 8\right) = \frac{55}{6}
\end{aligned}
$$

5. Compute the surface integral $\int_S x^2 + 2y^2\,dS$ over the surface

$$
S = \{(x,y,z) : x^2 + y^2 = z^2 \ \text{ and } \ 0 \le z \le 1\}.
$$

Suggestion: Use the parametrization $\mathbf{r} = \rho\cos\theta\mathbf{i} + \rho\sin\theta\mathbf{j} + \rho\mathbf{k}$ for the cone, with $0 \le \rho \le 1$ and $0 \le \theta < 2\pi$.

Solution: *With the suggested parametrization we have*

$$
\mathbf{r}_\rho = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \mathbf{k} \quad \textit{and} \quad \mathbf{r}_\theta = -\rho\sin\theta\mathbf{i} + \rho\cos\theta\mathbf{j},
$$

from which it follows that

$$
dS = \|\mathbf{r}_\rho \times \mathbf{r}_\theta\|\,dA = \|-\rho\cos\theta\mathbf{i} - \rho\sin\theta\mathbf{j} + \rho\mathbf{k}\|\,dA = \sqrt{2}\rho\,d\rho\,d\theta.
$$

Therefore,

$$
\begin{aligned}
\int_S x^2 + 2y^2\,dS &= \int_0^{2\pi}\int_0^1 \left(\rho^2\cos^2\theta + 2\rho^2\sin^2\theta\right)\sqrt{2}\rho\,d\rho\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}(1+\sin^2\theta)\int_0^1 \rho^3\,d\rho\,d\theta \\
&= \frac{\sqrt{2}}{4}\int_0^{2\pi} 1 + \sin^2\theta\,d\theta = \frac{3\sqrt{2}\pi}{4}
\end{aligned}
$$

6. If $\varphi(x,y,z)$ is a scalar field and $\mathbf{v}(x,y,z)$ is a vector field, show that

$$
\nabla \times (\nabla\varphi) = 0 \ \text{ and } \ \nabla \cdot (\nabla \times \mathbf{v}) = 0.
$$

Solution: *For the scalar field φ, we have*

$$
\nabla \cdot \varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k},
$$

so, assuming that all the second order derivatives are continuous, so that $\varphi_{xy} = \varphi_{yx}$, $\varphi_{xz} = \varphi_{zx}$ and $\varphi_{yz} = \varphi_{zy}$, it follows that

$$
\nabla \times (\nabla \cdot \varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = (\varphi_{zy} - \varphi_{yz})\mathbf{i} + (\varphi_{xz} - \varphi_{zx})\mathbf{j} + (\varphi_{yx} - \varphi_{xy})\mathbf{k} = \mathbf{0}.
$$

For the vector field $\mathbf{v} = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$, *we have*

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

Therefore (once again assuming that all second order partial derivatives are continuous)

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y} = 0.$$

7. Suppose that $\mathbf{u} = (\phi(x,y), \psi(x,y), 0)$ is a continuous vector field confined to $\mathbb{R}^2$ that is both *solenoidal* and *irrotational*. Show that the functions ϕ and ψ are both *harmonic* (potential) functions (i.e., they are each solutions of Laplace's equation).

Solution. If $\mathbf{u}$ is solenoidal, then its *divergence* is 0, i.e.,

$$\nabla \cdot \mathbf{u} = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 0,$$

so $\psi_y = -\phi_x$. If $\mathbf{u}$ is irrotational, then its *curl* is $\mathbf{0}$, i.e.,

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & 0 \end{vmatrix} = \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\mathbf{k} = \mathbf{0},$$

so $\psi_x = \phi_y$. It follows that the function $f(x+iy) = \psi(x,y) + i\phi(x,y)$ satisfies the Cauchy-Riemann equations and is therefore analytic in $\mathbb{C}$.

8. A (left) stochastic matrix $\mathbf{A}$ is an $n \times n$ matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

with nonnegative coefficients whose column-sums are all 1, i.e., for which $\sum_{i=1}^{n} a_{ij} = 1$ for each j.

(a) Show that a stochastic matrix always has an eigenvalue equal to 1.

Proof 1. *If* $\mathbf{I}$ *is the* $n \times n$ *identity matrix, then the column-sums of* $\mathbf{A} - \mathbf{I}$ *are all 0, because the sum of the entries in the* j^{th} *column of* $\mathbf{A} - \mathbf{I}$ *is*

$$a_{1j} + a_{2j} + \cdots + (a_{jj} - 1) + \cdots + a_{nj} = (a_{1j} + \cdots + a_{nj}) - 1 = 0.$$

It follows that the sum of the rows of $\mathbf{A} - \mathbf{I}$ *is the zero (row) vector, i.e., the rows of* $\mathbf{A} - \mathbf{I}$ *are linearly* ***dependent****, so rank*$(\mathbf{A} - \mathbf{I}) \leq n - 1$*. This implies that* $\det(\mathbf{A} - \mathbf{I}) = 0$*, so* $\lambda = 1$ *is an eigenvalue of* $\mathbf{A}$.

Proof 2: *If* $\mathbf{x} = [1\ 1\ \cdots\ 1]^T$*, then*

$$\mathbf{A}^T\mathbf{x} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

i.e., $\mathbf{A}^T\mathbf{x} = \mathbf{x}$*, which means that* $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}^T$ *with eigenvalue* $\lambda = 1$*. But the eigenvalues of* $\mathbf{A}$ *and* $\mathbf{A}^T$ *are the same, so* $\lambda = 1$ *is an eigenvalue of* $\mathbf{A}$ *too.*

(b) Find the eigenvalues and corresponding eigenvectors of the *stochastic* matrix

$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{bmatrix}.$$

Solution. *Eigenvalues:*

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 0.7-\lambda & 0.8 \\ 0.3 & 0.2-\lambda \end{vmatrix} = \lambda^2 - 0.9\lambda - 0.1 = (\lambda-1)(\lambda+0.1),$$

so the eigenvalues of $\mathbf{A}$ *are* $\lambda_1 = 1$ *and* $\lambda_2 = -0.1$.
Eigenvectors: we solve the equations $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$ *and* $(\mathbf{A} + 0.1\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\lambda_1 = 1: \qquad \begin{bmatrix} -0.3 & 0.8 \\ 0.3 & -0.8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -0.3x + 0.8y = 0 \implies y = \frac{3}{8}x,$$

so $\mathbf{x}_1 = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ *is an eigenvector with eigenvalue* $\lambda_1 = 1$.

$$\lambda_2 = -0.1: \qquad \begin{bmatrix} 0.8 & 0.8 \\ 0.3 & 0.3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x + y = 0 \implies y = -x,$$

so $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *is an eigenvector with eigenvalue* $\lambda_2 = -0.1$.

(c) Show that if $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, then there is a vector $\mathbf{w}$ *that depends only on* $a+b$ such that $\mathbf{A}^n\mathbf{u} \to \mathbf{w}$.

Solution. *The eigenvectors of* $\mathbf{A}$ *are linearly independent, so for any* $\mathbf{u}$, *we have*

$$\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix} = c_1\begin{bmatrix} 8 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

From this it follows that

$$\mathbf{A}^n\mathbf{u} = c_1\mathbf{A}^n\begin{bmatrix} 8 \\ 3 \end{bmatrix} + c_2\mathbf{A}^n\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 8 \\ 3 \end{bmatrix} + c_2(-0.1)^n\begin{bmatrix} 1 \\ -1 \end{bmatrix} \to c_1\begin{bmatrix} 8 \\ 3 \end{bmatrix},$$

where

$$c_1 = \frac{\begin{vmatrix} a & 1 \\ b & -1 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{a+b}{11}.$$

In other words,

$$\mathbf{A}^n\mathbf{u} \longrightarrow (a+b)\begin{bmatrix} 8/11 \\ 3/11 \end{bmatrix}.$$

9. In a large forest, foxes prey on rabbits while the rabbits feed on the (unlimited) vegetation. The change over time of the fox and rabbit populations in this forest is modeled by the following linear system:

$$\begin{bmatrix} F_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -p & 1.2 \end{bmatrix} \cdot \begin{bmatrix} F_k \\ R_k \end{bmatrix},$$

where F_k is the size of the fox population in year k, R_k is the size of the rabbit population in year k and p is a positive number called the *predation parameter*, that accounts for deaths in the rabbit population due to predation by foxes. The matrix, T_p, on the right hand side of the equation is called the *transition matrix* of the model.

(i) Find the eigenvalues and corresponding eigenvectors for the transition matrix when the predation parameter is $p = 0.275$.

When $p = 0.275$, *the transition matrix is* $T_p = \begin{bmatrix} 0.5 & 0.3 \\ -0.275 & 1.2 \end{bmatrix}$ *and the characteristic equation of* T_p *is*

$$\begin{vmatrix} 0.5-\lambda & 0.3 \\ -0.275 & 1.2-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad \lambda^2 - 1.7\lambda + 0.6825 = 0,$$

whose roots are

$$\lambda_1 = \frac{1.7 + \sqrt{1.7^2 - 4 \cdot 0.6825}}{2} = 1.05 \quad \textit{and} \quad \lambda_2 = \frac{1.7 - \sqrt{1.7^2 - 4 \cdot 0.6825}}{2} = 0.65.$$

Next, we find eigenvectors by finding (nonzero) solutions to the systems $(T - \lambda_1 I)\mathbf{x} = \mathbf{0}$ *and* $(T - \lambda_2 I)\mathbf{x} = \mathbf{0}$.
For $\lambda_1 = 1.05$, *we have the system*

$$\begin{bmatrix} -0.55 & 0.3 \\ -0.275 & 0.15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This reduces to the single equation $-0.55x + 0.3y = 0$, *of which* $x = 6$ *and* $y = 11$ *is a solution, so*

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

is an eigenvector for λ_1.
For $\lambda_2 = 0.65$, *we have the system*

$$\begin{bmatrix} -0.15 & 0.3 \\ -0.275 & 0.55 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This reduces to the single equation $-0.15x + 0.3y = 0$, *of which* $x = 2$ *and* $y = 1$ *is a solution, so*

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is an eigenvector for λ_2.

(ii) If $F_0 = 4$, $R_0 = 20$ and $p = 0.275$, what can you say about the limit $\lim_{k\to\infty} \frac{R_k}{F_k}$?

First, we express $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 20 \end{bmatrix}$ *as a linear combination of* $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 4 \\ 20 \end{bmatrix} = c_1 \begin{bmatrix} 6 \\ 11 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 11 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 20 \end{bmatrix} = \begin{bmatrix} 2.25 \\ -4.75 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} F_k \\ R_k \end{bmatrix} = T^k \cdot \mathbf{x}_0 = 2.25 T^k \cdot \mathbf{v}_1 - 4.75 T^k \cdot \mathbf{v}_2 = 2.25 \cdot 1.05^k \begin{bmatrix} 6 \\ 11 \end{bmatrix} - 4.75 \cdot 0.65^k \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and so when k is large

$$\begin{bmatrix} F_k \\ R_k \end{bmatrix} \approx 2.25 \cdot 1.05^k \begin{bmatrix} 6 \\ 11 \end{bmatrix},$$

because 0.65^k approaches 0 (rapidly) as k grows large. Therefore

$$\lim_{k\to\infty} \frac{R_k}{F_k} = \frac{11}{6}.$$

(iii) With $p = 0.275$, find the *critical ratio* ρ^* such that if $R_0/F_0 > \rho^*$, then both populations survive, and if $R_0/F_0 \le \rho^*$, then both populations die off. Explain your work.

Repeating the previous argument for the initial population vector $\mathbf{x}_0 = \begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$, we have $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and therefore

$$\mathbf{x}_k = T_p^k \mathbf{x}_0 = c_1 T_p^k \mathbf{v}_1 + c_2 T_p^k \mathbf{v}_2 = c_1(1.05)^k \mathbf{v}_1 + c_2(0.65)^k \mathbf{v}_2.$$

Now, since $\lim_{k\to\infty} 0.65^k = 0$, it follows that the populations survive if and only if $c_1 > 0$. To see how c_1 depends on $\mathbf{x}_0$, I will use Cramer's rule:

$$c_1 = \frac{\begin{vmatrix} F_0 & 2 \\ R_0 & 1 \end{vmatrix}}{\begin{vmatrix} 6 & 2 \\ 11 & 1 \end{vmatrix}} = \frac{F_0 - 2R_0}{-16} = \frac{2R_0 - F_0}{16}.$$

From this it follows that $c_1 > 0$ if and only if $2R_0 > F_0$. In other words, for the populations to survive, the ratio R_0/F_0 must be bigger than $1/2$, i.e., $\rho^* = 1/2$.

(iv) Show that if (the predation parameter) $49/120 > p > 1/3$, then both populations die off (rapidly) if $F_0 > 0$, regardless of R_0. What happens when $F_0 = 0$?

The characteristic polynomial of T_p is

$$\varphi(\lambda) = \det(T_p - \lambda I) = \begin{vmatrix} 0.5 & 0.3 \\ -p & 1.2 \end{vmatrix} = \lambda^2 - 1.7\lambda + (0.6 + 0.3p),$$

so eigenvalues of T_p are

$$\lambda_1(p) = \frac{1.7 + \sqrt{2.89 - 4(0.6 + 0.3p)}}{2} = 0.85 + \frac{1}{2}\sqrt{0.49 - 1.2p}$$

and

$$\lambda_2(p) = \frac{1.7 - \sqrt{2.89 - 4(0.6 + 0.3p)}}{2} = 0.85 - \frac{1}{2}\sqrt{0.49 - 1.2p}.$$

If $49/120 > p > 1/3$, then

$$0 < 0.49 - 1.2p < 0.49 - 0.4 = 0.09$$

and therefore

$$0 < \lambda_2(p) < \lambda_1(p) < 0.85 + \frac{1}{2}\sqrt{0.09} = 1.$$

If $\mathbf{x}_1(p)$ *and* $\mathbf{x}_2(p)$ *are eigenvectors for* $\lambda_1(p)$ *and* $\lambda_2(p)$*, respectively, then they are linearly independent (because* $\lambda_1(p) \neq \lambda_2(p)$*), and we can write the initial population vector as a linear combination:*

$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \mathbf{u}_0 = c_1\mathbf{x}_1(p) + c_2\mathbf{x}_2(p).$$

Repeating the analysis of part (iii) shows that

$$\mathbf{x}_k = T_p^k\mathbf{x}_0 = c_1\lambda_1(p)^k\mathbf{x}_1 + c_2\lambda_2(p)^k\mathbf{x}_2 \longrightarrow \mathbf{0},$$

because $\lambda_1(p)^k \to 0$ *and* $\lambda_2(p)^k \to 0$.

If $F_0 = 0$ *(no foxes), then the rabbit population grows according to the simpler model* $R_{k+1} = 1.2R_k$*, or* $R_k = (1.2)^k R_0$*. I.e., the two-population (predator-prey) model doesn't apply and the rabbit population grows exponentially.*

Comments: *It is intuitively obvious that the larger the predation parameter, the less likely the populations will survive. So what happens when* $p \geq 49/120$*?*

(a) If $p = 49/120$*, then* T_p *has only one eigenvalue (*$\lambda_1 = \lambda_2 = 0.85$*) and is not diagonalizable. But it can be shown that in this case* $T_p = BUB^{-1}$*, where*

$$U = \begin{bmatrix} 0.85 & \alpha(p) \\ 0 & 0.85 \end{bmatrix},$$

for some real number $\alpha(p)$*, from which it follows (with a little more work) that*

$$T_p^k\mathbf{x} = BU^kB^{-1}\mathbf{x} \longrightarrow \mathbf{0}$$

for any $\mathbf{x} \in \mathbb{R}^2$ *in this case too.*

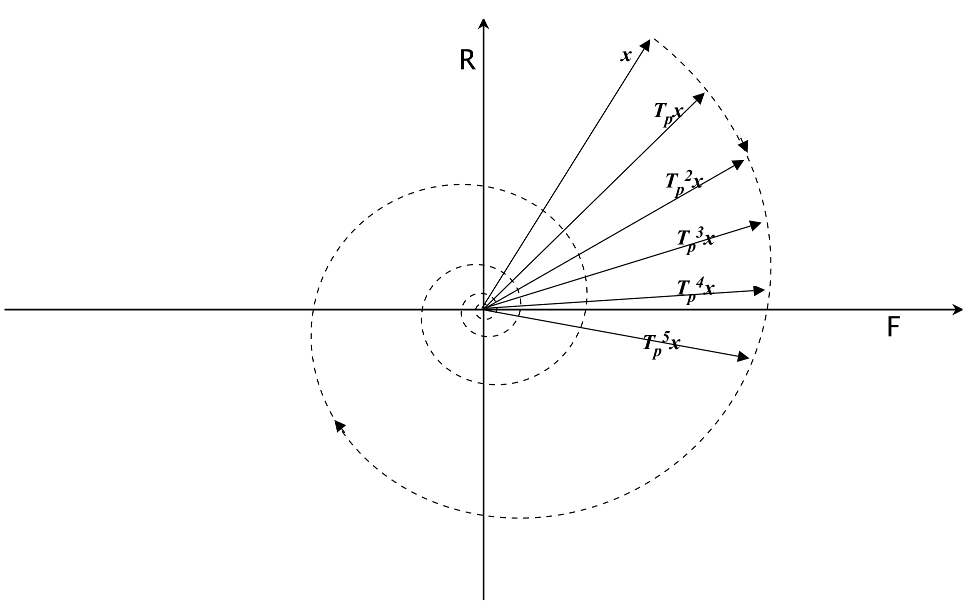


Figure 2: Orbit of population vector $\mathbf{x}_k = T_p^k\mathbf{x}$ when $p > 49/120$.

(b) If $p > 49/120$, then the eigenvalues of T_p are complex conjugates

$$\lambda_1 = 0.85 + i\beta(p) \;\; \textit{and} \;\; \lambda_2 = 0.85 - i\beta(p),$$

where $\beta(p) = \sqrt{1.2p - 0.49}$ is real and positive. In this case the orbit

$$\{T_p^k \mathbf{x} : k = 1, 2, 3, \ldots\}$$

***spirals in** towards $\mathbf{0}$ for every $\mathbf{x} \in \mathbb{R}^2$, so both populations die off in this case as well. In fact, as Figure 1 illustrates, the rabbit population dies off after a finite number of generations (somewhere between 4 and 5 in the figure), and with no rabbits, the fox population then declines according to the simpler model $F_{k+1} = 0.5F_k$.*

10. Consider the function $f(x) = e^x$ defined on the interval $[0, 1]$.

(a) Sketch the graph of its periodic extension to $\mathbb{R}$ with period 1, as well as its even and odd periodic extensions to $\mathbb{R}$ with period 2.

Solution: *The periodic extension of period 1 is the function $f_1(x) = e^x$ for $0 < x \le 1$, then continued periodically so that for every integer n, if $n < x \le n + 1$, then $f_1(x) = e^{x-n}$, see Figure 8 for its graph.*

The even periodic extension of period 2 is the function $f_{2e}(x)$, defined by $f_{2e}(x) = e^x$ for $0 < x \le 1$, $f_{2e}(x) = e^{-x}$ for $-1 < x \le 0$, and then continued periodically, so that for every integer n, if $2n - 1 < x \le 2n$, then

$$f_{2e}(x) = \begin{cases} e^{x-2n} & : \;\; 2n < x \le 2n + 1 \\ e^{2n-x} & : \;\; 2n - 1 < x \le 2n \end{cases}$$

Its graph is displayed in Figure 4.

The odd periodic extension of period 2 is the function $f_{2o}(x)$, defined by $f_{2o}(x) = e^x$ for $0 < x \le 1$, $f_{2o}(x) = -e^{-x}$ for $-1 < x \le 0$, and then continued periodically, so that for every integer n, if $2n - 1 < x \le 2n$, then

$$f_{2o}(x) = \begin{cases} e^{x-2n} & : \;\; 2n < x \le 2n + 1 \\ -e^{2n-x} & : \;\; 2n - 1 < x \le 2n \end{cases}$$

Its graph is displayed in Figure 5.

(b) Which of these periodic extensions will yield the *best* Fourier series expansion? Why?

Solution: *The even periodic extension should produce the most quickly converging Fourier series expansion because it is the only one that yields a continuous function.*

(c) Compute the Fourier coefficients for all three periodic extensions. Were you right?

Solution:

(i) Period 1 extension: *Denoting the coefficients of $\cos(2\pi nx)$ and $\sin(2\pi nx)$ by a_n and b_n respectively, we have*

$$\begin{aligned} a_n &= 2\int_0^1 e^x \cos(2\pi nx)\, dx \\ &= 2e^x \cos(2\pi nx)\Big|_0^1 + 4\pi n \int_0^1 e^x \sin(2\pi nx)\, dx \\ &= 2(e-1) + 4\pi n e^x \sin(2\pi nx)\Big|_0^1 - 8\pi^2 n^2 \int_0^1 e^x \cos(2\pi nx)\, dx \\ &= 2(e-1) - 4\pi^2 n^2 a_n \end{aligned}$$

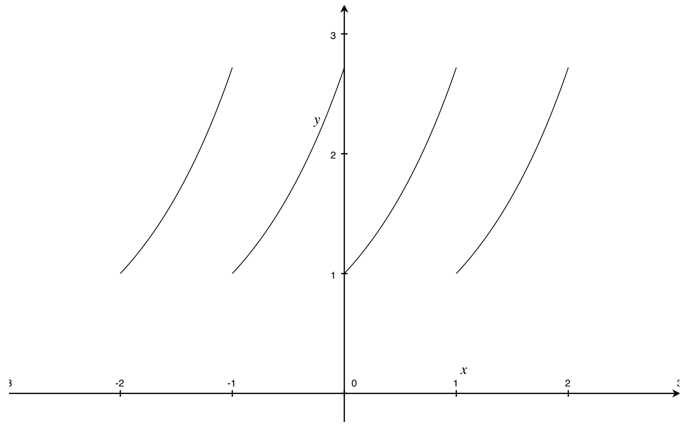


Figure 3: Periodic extension with period 1: $y = f_1(x)$

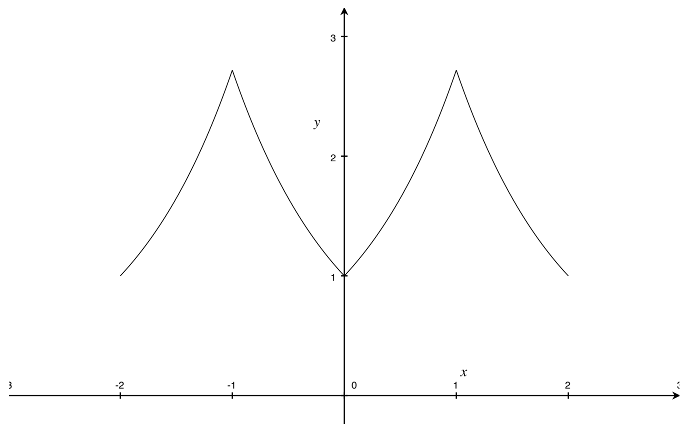


Figure 4: Even periodic extension with period 2: $y = f_{2e}(x)$

so

$$(1 + 4\pi^2 n^2)a_n = 2(e-1) \implies a_n = \frac{2(e-1)}{1+4\pi^2 n^2}.$$

Likewise,

$$\begin{aligned} b_n &= 2\int_0^1 e^x \sin(2\pi n x)\, dx \\ &= \ 2e^x \sin(2\pi n x)\Big|_0^1 - 4\pi n \int_0^1 e^x \cos(2\pi n x)\, dx \\ &= -2\pi n a_n, \end{aligned}$$

so

$$b_n = -\frac{4\pi n(e-1)}{1+4\pi^2 n^2}.$$

Thus the Fourier series for f_{1e} is

$$f_1(x) = (e-1)\left(1 + \sum_{n=1}^{\infty} \frac{2\cos(2\pi n x)}{4\pi^2 n^2 + 1} - \frac{4\pi n \sin(2\pi n x)}{4\pi^2 n^2 + 1}\right)$$

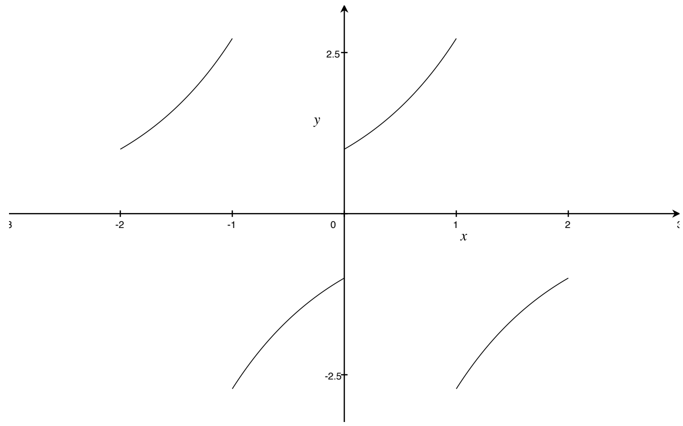


Figure 5: Odd periodic extension with period 2: $y = f_{2o}(x)$

In the figure below, the graph of f_1 *is displayed together with the truncation*

$$(e-1)\left(1+\sum_{n=1}^{7}\frac{2\cos(2\pi nx)}{4\pi^2n^2+1}-\frac{4\pi n\sin(2\pi nx)}{4\pi^2n^2+1}\right)$$

of its Fourier series (dashed red line).

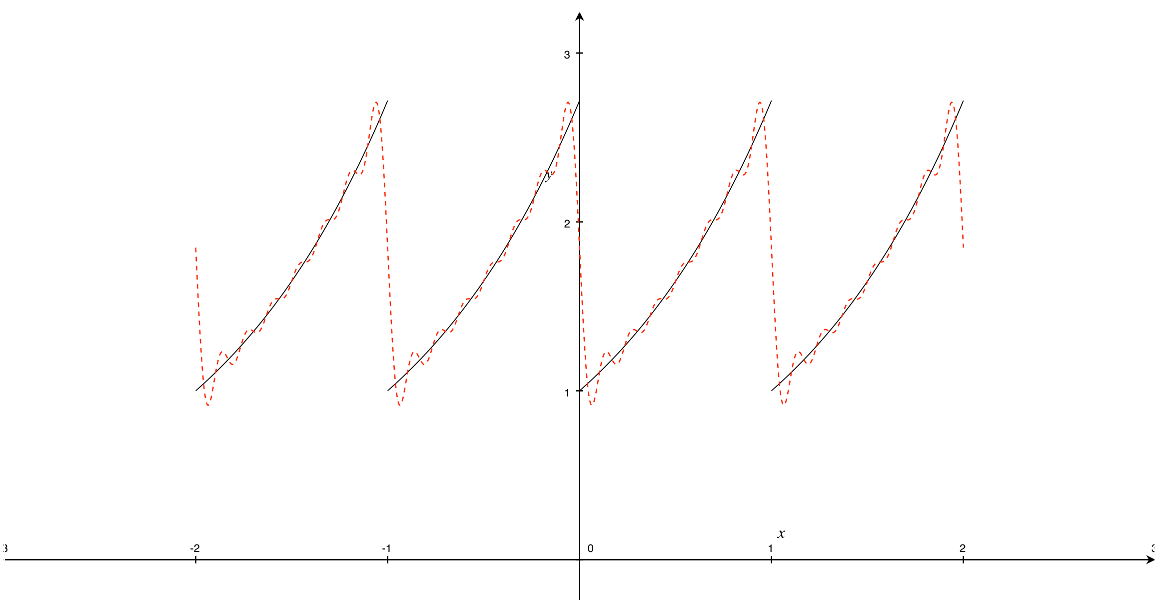


Figure 6: Approximation of $y = f_1(x)$, truncating its Fourier series at $n = 7$.

(ii) Even extension *(period 2): Since* f_{2e} *is an even function, its Fourier series will consist* *only of cosine terms. Denoting the coefficients by a_n again, we have*

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f_{2e}(x)\cos(\pi n x)\,dx = 2\int_0^1 e^x \cos(\pi n x)\,dx \qquad \text{(because } f_{2e}(x)\cos(\pi n x) \text{ is even).}\\
&= 2e^x\cos(\pi n x)\Big|_0^1 + 2\pi n\int_0^1 e^x \sin(\pi n x)\,dx\\
&= 2(e-1) + 2\pi n e^x \sin(\pi n x)\Big|_0^1 - 2\pi^2 n^2 \int_0^1 e^x\cos(\pi n x)\,dx\\
&= 2((-1)^n e - 1) - \pi^2 n^2 a_n.
\end{aligned}
$$

Hence

$$a_n = \frac{2((-1)^n e - 1)}{\pi^2 n^2 + 1},$$

and therefore

$$f_{2e}(x) = (e-1) + 2\sum_{n=1}^{\infty} \frac{((-1)^n e - 1)\cos(\pi n x)}{\pi^2 n^2 + 1}.$$

In the figure below, the graph of f_{2e} is displayed together with the truncation

$$(e-1) + 2\sum_{n=1}^{7} \frac{((-1)^n e - 1)\cos(\pi n x)}{\pi^2 n^2 + 1}$$

of its Fourier series (dashed red line).

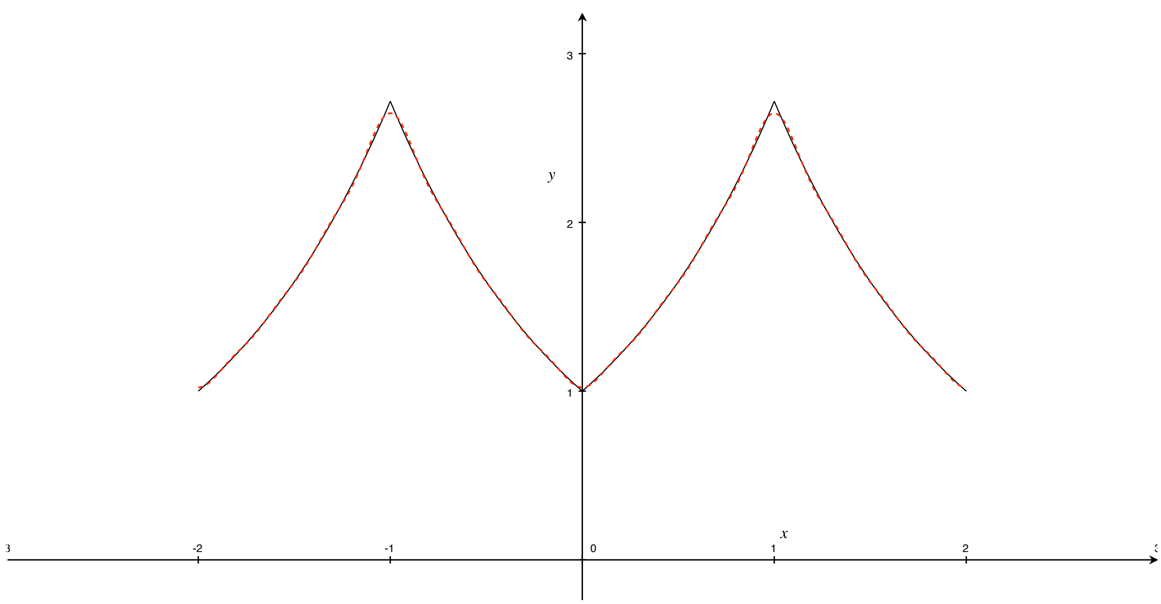


Figure 7: Approximation of $y = f_{2e}(x)$, truncating its Fourier series at $n = 7$.

(iii) Odd extension *(period 2): Since f_{2o} is an even function, its Fourier series will consist only of sine terms. Denoting the sine coefficients by b_n again, and using the fact that the product*

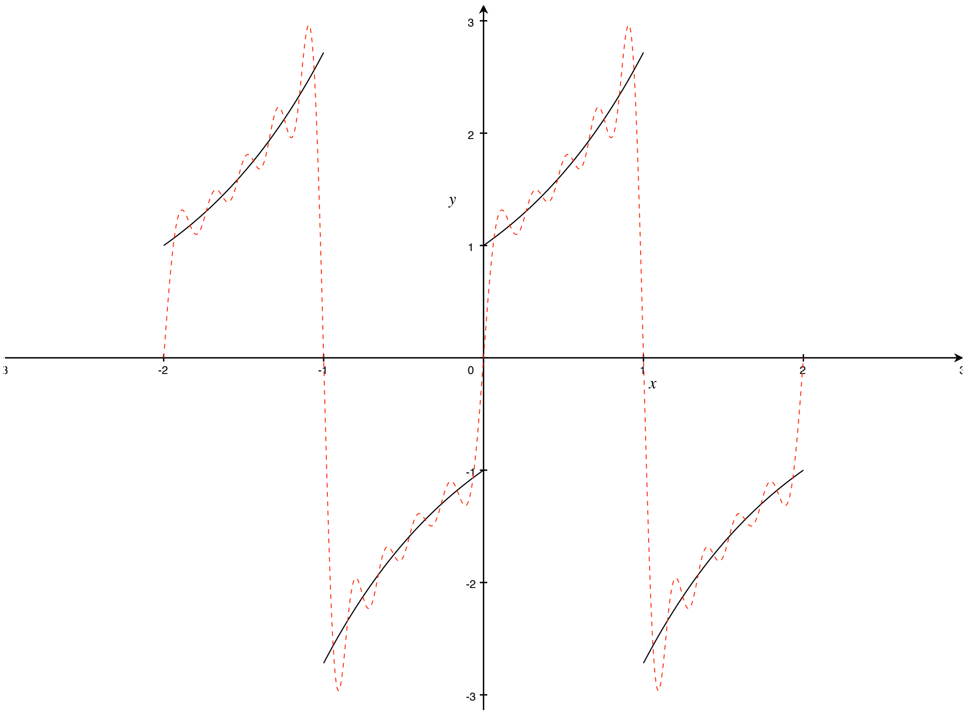


Figure 8: Approximation of $y = f_{2o}(x)$, truncating its Fourier series at $n = 10$.

of odd functions is even, we have

$$\begin{aligned} b_n &= \int_{-1}^{1} f_{2o}(x)\sin(\pi n x)\,dx = 2\int_0^1 e^x \sin(\pi n x)\,dx \qquad \text{(because } f_{2o}(x)\sin(\pi n x) \text{ is } \textbf{\textit{even}}\text{).} \\ &= \ 2e^x \sin(\pi n x)\Big|_0^1 - 2\pi n \int_0^1 e^x \cos(\pi n x)\,dx \\ &= -\ 2\pi n e^x \cos(\pi n x)\Big|_0^1 - 2\pi^2 n^2 \int_0^1 e^x \sin(\pi n x)\,dx \\ &= 2\pi n((-1)^{n+1}e + 1) - \pi^2 n^2 b_n. \end{aligned}$$

Hence

$$b_n = \frac{2\pi n((-1)^{n+1}e + 1)}{\pi^2 n^2 + 1},$$

and therefore

$$f_{2e}(x) = 2\pi \sum_{n=1}^{\infty} \frac{n((-1)^{n+1}e + 1)\sin(\pi n x)}{\pi^2 n^2 + 1}.$$

In the figure below, the graph of f_{2o} is displayed together with the truncation of its Fourier series at $n = 10$ (dashed red line).

The coefficients of the even periodic extension are all on the order of $1/n^2$, while the other two periodic extensions have coefficients on the order of $1/n$, so, yes, I was right. This is also visible in the graphs — the truncated Fourier series of the even periodic extension yields a much better approximation of f_{2e} than the other two truncated series do for their respective functions.

11. Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$ be a non constant polynomial with complex coefficients (i.e., $n > 0$ and $a_n \neq 0$). Use Rouché's theorem to show that $p(z)$ has exactly n roots in $\mathbb{C}$ (counting multiplicity).

Solution. *The idea is to show that if $|z| = R$ is large enough, then*

$$|a_n z^n| > \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right|.$$

It will then follow from Rouché's theorem that $a_n z^n$ and $p(z) = a_n z^n + (a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$ have the same number of zeros (namely n) inside the disk $\{z \in \mathbb{C} : |z| < R\}$.

To find a sufficiently large R, first observe that if $|z| > 1$, then by the triangle inequality

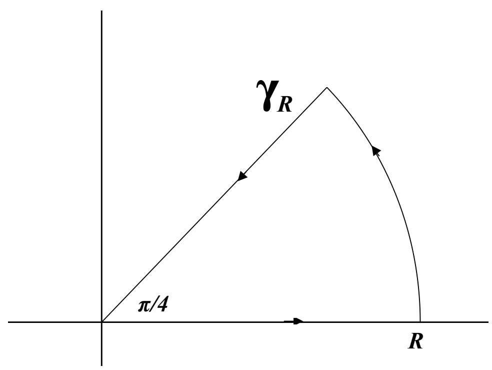
$$\begin{aligned}\left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| &= |z|^n \left|a_{n-1}z^{-1} + \cdots + a_1 z^{1-n} + a_0 z^{-n}\right| \\ &\leq |z|^n \left(|a_{n-1}||z|^{-1} + \cdots + |a_1||z|^{1-n} + |a_0||z|^{-n}\right) \\ &\leq |z|^n \left(|a_{n-1}|z|^{-1} + \cdots + |a_1||z|^{-1} + |a_0||z|^{-1}\right) \leq |z|^n \cdot \left(nM|z|^{-1}\right),\end{aligned}$$

where $M = \max\limits_{0 \leq j \leq n-1} |a_j|$. It follows that if $|z| = R = \dfrac{nM}{|a_n|} + 1$, then $|z| > 1$ and $nM|z|^{-1} < |a_n|$, from which it follows that

$$\left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| \leq |z|^n \cdot \left(nM|z|^{-1}\right) < |a_n z^n|.$$

12. Use Cauchy's theorem and the countours γ_r (illustrated below) in $\mathbb{C}$ to show that

$$\int_0^\infty \cos(\alpha x^2)\, dx = \sqrt{\frac{\pi}{8\alpha}}.$$



The function $\exp(i\alpha z^2)$ is holomorphic in all of $\mathbb{C}$ (since $\frac{d}{dx}\exp(i\alpha z^2) = 2i\alpha z\exp(i\alpha z^2)$ exists in all of $\mathbb{C}$). This means that if γ is any closed curve in $\mathbb{C}$, then by Cauchy's theorem,

$$\oint_\gamma \exp(i\alpha z^2)\, dz = 0.$$

If $z = x$ is real, then $\exp(i\alpha x^2) = \cos(\alpha x^2) + i\sin(\alpha x^2)$, thus the integral $\int_0^\infty \cos(\alpha x^2)\, dx$ is the real part of the integral $\int_0^\infty \exp(i\alpha x^2)\, dx$.

The idea then, is to find a closed contour γ that includes the positive real axis such that that the integral of $\exp(i\alpha z^2)\, dz$ along the remainder of γ can be computed directly.

Observation 1: *If we write $z = r(\cos\theta + i\sin\theta)$, then*

$$\exp(i\alpha z^2) = \exp\left(i\alpha r^2(\cos 2\theta + i\sin 2\theta)\right) = e^{-\alpha r^2 \sin 2\theta} e^{i\alpha r^2 \cos 2\theta}.$$

Now, if $0 < \theta < \pi/2$, then $\sin 2\theta > 0$ and $|\exp(i\alpha z^2)| = e^{-\alpha r^2 \sin 2\theta}$ which goes to 0 as $r \to \infty$, ***for fixed $\boldsymbol{\theta}$***. *This would appear to indicate that*

$$\lim_{r\to\infty} \int_{\gamma_1(r)} \exp(i\alpha z^2)\, dz = 0,$$

where $\gamma_1(r)$ is the eighth of a circle in the first quadrant of radius r with the counterclockwise orientation, depicted below. More on this later.

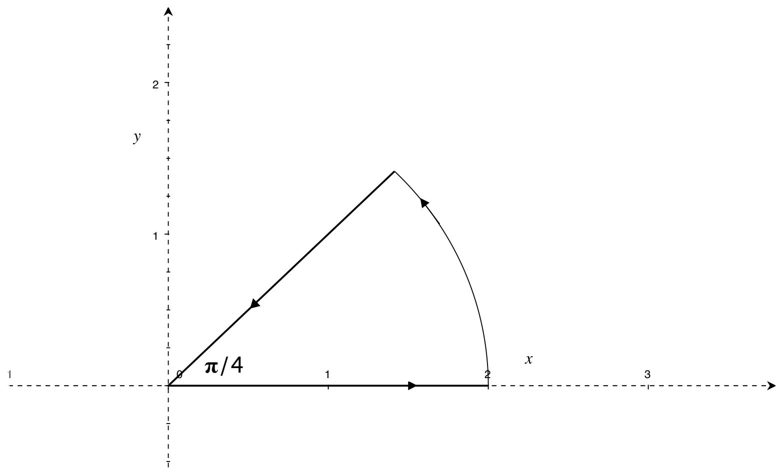


Figure 9: The curves $\gamma_1(r)$ and $\gamma_2(r)$.

Observation 2: *If $z = te^{i\pi/4}$, then $z^2 = t^2 e^{i\pi/2} = it^2$ so $\exp(i\alpha z^2) = e^{-\alpha t^2}$ in this case. It follows that if $\gamma_2(r)$ is the ray from 0 to $re^{i\pi/4}$(oriented towards 0 as depicted above), then*

$$\int_{\gamma_2(r)} \exp(i\alpha z^2)\, dz = \int_r^0 e^{-\alpha t^2} e^{i\pi/4}\, dt = -\frac{1+i}{\sqrt{2}} \int_0^r e^{-\alpha t^2}\, dt,$$

(since along $\gamma_2(r)$, $dz = e^{i\pi/4}\, dt$). Hence as $r \to \infty$, the real part of $\int_{\gamma_2(r)} \exp(i\alpha z^2)\, dz$ approaches the limit

$$-\frac{1}{\sqrt{2}} \int_0^\infty e^{-\alpha t^2}\, dt = -\frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{4\alpha}} = -\sqrt{\frac{\pi}{8\alpha}},$$

because

$$\int_0^\infty e^{-\alpha t^2}\, dt = \frac{1}{\sqrt{\alpha}} \int_0^\infty e^{-x^2}\, dx = \frac{1}{\sqrt{\alpha}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{4\alpha}}.$$

Conclusion:[‡] *Let γ_r be the closed curve comprised of the segment $[0, r]$ on the real line, $\gamma_1(r)$ and $\gamma_2(r)$ (with the standard counterclockwise orientation), then*

$$\oint_{\gamma_r} e^{i\alpha z^2}\, dz = \int_0^r e^{i\alpha x^2}\, dx + \int_{\gamma_1(r)} e^{i\alpha z^2}\, dz + \int_{\gamma_2(r)} e^{i\alpha z^2}\, dz = 0$$

from which it follows, upon comparing real parts, that

$$\int_0^r \cos(ax^2)\, dx = \Re\left(\int_0^r e^{i\alpha x^2}\, dx\right) = -\Re\left(\int_{\gamma_1(r)} e^{i\alpha z^2}\, dz + \int_{\gamma_2(r)} e^{i\alpha z^2}\, dz\right).$$

[‡]Except for the fun part.

Now, if it is indeed true that

$$\lim_{r\to\infty}\int_{\gamma_1(r)} e^{i\alpha z^2}\,dz = 0,$$

then it follows that

$$\int_0^\infty \cos(\alpha x^2)\,dx = \lim_{r\to\infty}\int_0^r \cos(\alpha x^2)\,dx = \lim_{r\to\infty} -\Re\left(\int_{\gamma_1(r)} e^{i\alpha z^2}\,dz + \int_{\gamma_2(r)} e^{i\alpha z^2}\,dz\right) = 0 + \sqrt{\frac{\pi}{8\alpha}},$$

as claimed.

The fun part: *We want to show that* $\int_{\gamma_1(r)} e^{i\alpha z^2}\,dz \to 0$ *as* $r\to\infty$, *which we do using the inequality*

$$\left|\int_\Gamma f(z)\,dz\right| \le \max_{z\in\Gamma}|f(z)|\cdot \ell(\Gamma),$$

where $\ell(\Gamma)$ *is the length of the curve* Γ. *In this case,* $\ell(\gamma_1(r)) = \pi r/4 < r$ *and* $|e^{i\alpha z^2}| = e^{-\alpha r^2\sin 2\theta}$, *and since for a constant* $c>0$, $xe^{-cx^2}\to 0$ *(rapidly) as* $x\to\infty$, *we would like to say that* $re^{-\alpha r^2\sin 2\theta}\to 0$ *(which implies that the integral goes to* 0*). But it's not quite that simple, because if for example* $\theta = 1/r^2$ *with* r *large (so* θ *is very close to* 0*) then* $\sin 2\theta \approx 2\theta = 2/r^2$, *in which case*

$$e^{-\alpha r^2\sin 2\theta} \approx e^{-\alpha r^2 2\theta} = e^{-2\alpha},$$

which is bounded away from 0. *On the other hand, this phenomenon only occurs on a* ***very*** *short portion of* $\gamma_1(r)$, *which suggests the following fix.*

For a (very small) angle $\theta_r > 0$,[§] *we divide* $\gamma_1(r)$ *into two parts,* $\Gamma_1(r)$ *and* $\Gamma_2(r)$, *where* Γ_1 *is the (very short) portion of* $\gamma_1(r)$ *where* $0\le\theta\le\theta_r$ *and* Γ_2 *is the portion of* $\gamma_1(r)$ *where* $\theta_r<\theta\le\pi/4$. *The lengths of these two parts are*

$$\ell(\Gamma_1(r)) = r\theta_r \quad \text{and} \quad \ell(\Gamma_2(r)) = r\left(\frac{\pi}{4}-\theta_r\right) < r.$$

Next, for any θ *between* 0 *and* $\pi/4$, $e^{-\alpha r^2\sin 2\theta}\le 1$ *(since* $\sin 2\theta\ge 0$*) and therefore*

$$\left|\int_{\Gamma_1(r)} e^{i\alpha z^2}\,dz\right| \le \ell(\Gamma_1(r))\cdot 1 = r\theta_r.$$

On the other hand, if $\pi/4\ge\theta>\theta_r>0$, *then* $\sin 2\theta > \sin 2\theta_r > 0$ *so*

$$e^{-\alpha r^2\sin 2\theta} < e^{-\alpha r^2\sin 2\theta_r}.$$

Furthermore, if $0<\theta_r<\pi/6$, *then* $\sin 2\theta_r > \theta_r$,[¶] *and it follows that if* $\theta_r<\pi/6$, *then*

$$\left|\int_{\Gamma_2(r)} e^{i\alpha z^2}\,dz\right| \le \ell(\Gamma_2(r))e^{-\alpha r^2\sin 2\theta_r} < re^{-\alpha r^2\sin 2\theta_r} < re^{-\alpha r^2\theta_r}.$$

Combining these two results, we have

$$\left|\int_{\gamma_1(r)} e^{i\alpha z^2}\,dz\right| = \left|\int_{\Gamma_1(r)} e^{i\alpha z^2}\,dz + \int_{\Gamma_2(r)} e^{i\alpha z^2}\,dz\right| \le \left|\int_{\Gamma_1(r)} e^{i\alpha z^2}\,dz\right| + \left|\int_{\Gamma_2(r)} e^{i\alpha z^2}\,dz\right| < r\theta_r + re^{-\alpha r^2\theta_r},$$

[§] Which, as indicated, we choose to depend on r.

[¶] Because: (i) $\sin 0 = 0$ and (ii) $(\sin 2x - x)' > 0$ for $0<x<\pi/6$. Fill in the details.

Finally, given $\varepsilon > 0$, if $\theta_r = \varepsilon/2r$, then

$$r\theta_r = \varepsilon/2 \quad \textit{and} \quad re^{-\alpha r^2\theta_r} = re^{-(\alpha\varepsilon/2)r}.$$

*Since $re^{-cr} \to 0$ as $r \to \infty$ for any $c > 0$, it follows that (for **fixed** ε) there is an R_ε such that if $r > R_\varepsilon$, then*

$$re^{-(a\varepsilon/2)r} < \frac{\varepsilon}{2}.$$

Hence, for $r > R_\varepsilon$

$$\left|\int_{\gamma_1(r)} e^{i\alpha z^2}\, dz\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and since ε was arbitrary, it follows that

$$\lim_{r\to\infty} \int_{\gamma_1(r)} e^{i\alpha z^2}\, dz = 0,$$

as claimed.